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Positive Strongly Decreasing Solutions of Emden-Fowler Type Second-Order Difference Equations with Regularly Varying Coefficients

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Abstract. Positive decreasing solutions of the nonlinear difference equation

$$\Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) = q_n|x_{n+1}|^{\beta-1}x_{n+1}, \quad n \geq 1, \quad \alpha > \beta > 0,$$

are studied under the assumption that p, q are regularly varying sequences. Necessary and sufficient conditions are established for the existence of regularly varying strongly decreasing solutions and it is shown that the asymptotic behavior of all such solutions is governed by a unique formula.

1. Introduction

Consider the nonlinear difference equation of second order

$$(E) \qquad \Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) = q_n|x_{n+1}|^{\beta-1}x_{n+1}, \quad n \geq 1,$$

where α and β are positive constants such that $\alpha > \beta$, $p = \{p_n\}$, $q = \{q_n\}$ are positive real sequences and Δ is forward difference operator defined as $\Delta x_n = x_{n+1} - x_n$. In our case, when $\alpha > \beta$, equation (E) is said to be sub-half-linear, while otherwise, for $\alpha = \beta$ or $\alpha < \beta$ equation (E) is called half-linear or super-half-linear, respectively.

By a solution of (E) we mean a not trivial real sequence $x = \{x_n\}$ satisfying (E). A solution x of the equation (E) is called oscillatory if for every $M \in \mathbb{N}$ there exist $m, n \in \mathbb{N}$, $M \leq m < n$ such that $x_m x_n < 0$, otherwise, it is called nonoscillatory. In other words, a solution x is called nonoscillatory if it is eventually positive or eventually negative. It is known that every solution of (E) is nonoscillatory. If $x = \{x_n\}$ is a solution of (E), then clearly $-x = \{-x_n\}$ is also a solution. Thus, in studying nonoscillatory solutions of (E), for the sake of simplicity, we restrict ourself to solutions which are eventually positive. Any such solution $\{x_n\}$ is eventually strongly monotone and belongs to one of the two classes listed below (see [6, Lemma 1]):

$$\begin{aligned} \mathbb{M}^+ &= \{x \text{ solution of } (E) \,|\, \exists\, n_0 \geq 1 \,:\, x_n > 0,\ \Delta x_n > 0,\ \text{for } n \geq n_0\}, \\ \mathbb{M}^- &= \{x \text{ solution of } (E) \,|\, x_n > 0,\ \Delta x_n < 0,\ \text{for } n \geq 1\}. \end{aligned}$$

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Positive strongly decreasing solutions of Emden-Fowler type second-order difference equations with regularly varying coefficients

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1 Introduction

Good morning. How are you doing? (Answer) OK, let's begin. Today, I am going to talk about Emden-Fowler type difference equation. This type of equation we have already observe, but in the continuous case. Now, we move to the discrete case. Why is this equation important? As we said in previous classes, differential equations play an important role in modeling almost all physical, technical or biological processes from celestial motion through bridge design to interaction between neurons. Differential equations such as those used to solve real-life problems do not necessarily have to be directly solvable. On the other hand, in the last fifty years, the application of difference equations in solving many problems in statistics, engineering and science in general has experienced expansion. The development of high-speed digital computer technology has motivated the application of difference equations to ordinary and partial differential equations. Apart from this, difference equations are very useful for analyzing electrical, mechanical, thermal and other systems, the behavior of electric-wave filters and other filters, insulator strings, crankshafts of multi-cylinder engines and so on. One of the most studied second-order nonlinear differential equations is

$$(1.1) \qquad \big(p(t)|x'|^{\alpha-1}x'\big) = q(t)|x|^{\beta-1}x, \quad \alpha, \beta > 0,$$

where p, q are continuous positive functions on $[a, \infty)$. Along with the differential equation (1.1), discrete counterpart of this equation

$$(E) \qquad \Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) = q_n|x_{n+1}|^{\beta-1}x_{n+1}, \quad n \geq 1,$$

where α and β are positive constants such that $\alpha > \beta$, $p = \{p_n\}$, $q = \{q_n\}$ are positive real sequences and Δ is forward difference operator defined as $\Delta x_n = x_{n+1} - x_n$ has attracted many researchers. Today, we will talk about the equation (E). We'll look at how to find necessary and sufficient conditions for the existence of regularly varying solutions. Also, our goal is to determine the asymptotic formula for all solutions of equation that we consider. I would like to stress that there are similarities between equations (1.1) and (E), but we will notice some differences as well. Let us start with some basic concepts. As you can see from (E), this equation is nonlinear (pointing to α and β) difference equation of second order (indicating that the operator Δ appears twice). We say that equation (E) is sub-half-linear, or just, sub-linear when $\alpha > \beta$ while otherwise, for $\alpha = \beta$ or $\alpha < \beta$ equation (E) is called half-linear or super-half-linear, respectively.
I said at the beginning that we will discuss about solutions of equation (E). What do we

It is well-known that the differential equation

$$\left(p(t)|x'|^{\alpha-1}x'\right)=q(t)|x|^{\beta-1}x,\quad \alpha,\beta>0\,, \tag{1.1}$$

where p,q are continuous positive functions on $[a,\infty)$, may have a nontrivial solution x, with the property that there exists $T_x<\infty$, such that $x(t)\equiv 0$ on $[T_x,\infty)$. Such a solution is said to be *extinct singular solution* or *singular solution of the first kind*. On the contrary, such solutions of difference equation (E) do not exists. One more difference between differential and difference equations is that for the differential equation (1.1) classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be empty, while for the difference equation (E), this case cannot occur (see [6] and [1, Section 5.3]). The asymptotic behaviour of nonoscillatory solutions for nonlinear second-order difference equations has been studied in many papers, see, e.g. [2, 3], [6]-[12], [27], [38], the monograph [1] and references therein.

For any solution x of (E) denote by $x^{[1]}=\left\{x_n^{[1]}\right\}$ its quasi-difference $x_n^{[1]}=p_n|\Delta x_n|^{\alpha-1}\Delta x_n$. Thus, under our assumptions, the classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be a-priori divided into the following subclasses:

$$\begin{aligned}
\mathbb{M}^+_{\infty,\infty} &= \{x\in\mathbb{M}^+ : \lim_n x_n=\infty,\ \lim_n x_n^{[1]}=\infty,\}\,,\\
\mathbb{M}^+_{\infty,l} &= \{x\in\mathbb{M}^+ : \lim_n x_n=\infty,\ \lim_n x_n^{[1]}=l,\ 0<l<\infty\}\,,\\
\mathbb{M}^+_{k,\infty} &= \{x\in\mathbb{M}^+ : \lim_n x_n=k,\ \ 0<k<\infty,\ \lim_n x_n^{[1]}=\infty\}\,,\\
\mathbb{M}^+_{k,l} &= \{x\in\mathbb{M}^+ : \lim_n x_n=k,\ \ 0<k<\infty,\ \lim_n x_n^{[1]}=l,\ 0<l<\infty\}\,,\\
\mathbb{M}^-_{k,l} &= \{x\in\mathbb{M}^- : \lim_n x_n=k,\ \ 0<k<\infty,\ \lim_n x_n^{[1]}=-l,\ 0<l<\infty\}\,,\\
\mathbb{M}^-_{0,l} &= \{x\in\mathbb{M}^- : \lim_n x_n=0,\ \lim_n x_n^{[1]}=-l,\ \ 0<l<\infty\}\\
\mathbb{M}^-_{k,0} &= \{x\in\mathbb{M}^- : \lim_n x_n=k,\ \ 0<k<\infty,\ \lim_n x_n^{[1]}=0\}\,,\\
\mathbb{M}^-_{0,0} &= \{x\in\mathbb{M}^- : \lim_n x_n=0,\ \lim_n x_n^{[1]}=0\}\,.
\end{aligned}$$

A solution $x\in\mathbb{M}^+_{\infty,\infty}$ is said to be *strongly increasing* and a solution $x\in\mathbb{M}^-_{0,0}$ is said to be *strongly decreasing* or *strongly decaying*. For solutions which tends to some constant we use $\mathbb{M}^-_B=\mathbb{M}^-_{k,0}\cup\mathbb{M}^-_{k,l}$, $\mathbb{M}^+_B=\mathbb{M}^+_{k,\infty}\cup\mathbb{M}^+_{k,l}$ and for decreasing solutions which tends to zero we use $\mathbb{M}^-_0=\mathbb{M}^-_{0,l}\cup\mathbb{M}^-_{0,0}$.

Let

$$S=\sum_{n=1}^{\infty}\frac{1}{p_n^{1/\alpha}}.$$

Depending on whether $S=\infty$ or $S<\infty$ some of the above classes may be empty.

(i) If $S=\infty$ then

$$\mathbb{M}^+=\mathbb{M}^+_{\infty,\infty}\cup\mathbb{M}^+_{\infty,l}\quad\text{and}\quad \mathbb{M}^-=\mathbb{M}^-_{k,0}\cup\mathbb{M}^-_{0,0},\quad\text{i.e.}\quad \mathbb{M}^+_B=\emptyset\,,\ \mathbb{M}^-_{0,l}\cup\mathbb{M}^-_{k,l}=\emptyset.$$

(ii) If $S<\infty$ then

$$\mathbb{M}^+=\mathbb{M}^+_{\infty,\infty}\cup\mathbb{M}^+_B\quad\text{and}\quad \mathbb{M}^-=\mathbb{M}^-_0\cup\mathbb{M}^-_B,\quad\text{i.e.}\quad \mathbb{M}^+_{\infty,l}=\emptyset.$$

In this paper, we consider only positive decreasing solutions, i.e. solutions in $\mathbb{M}^-$. Concerning the existence of solutions in the classes $\mathbb{M}^-_B$ and $\mathbb{M}^-_{0,l}$, the following holds.

Theorem 1.1. *(i) Equation (E) has solutions in $\mathbb{M}^-_B$ if and only if*

$$I=\sum_{n=1}^{\infty}\left(\frac{1}{p_n}\sum_{k=n}^{\infty}q_k\right)^{\frac{1}{\alpha}}<\infty.$$

mean when we say that x is the solution of the equation (E)? By a solution of (E) we mean a not trivial real sequence $x=\{x_n\}$ satisfying (E). Which means not trivial? A solution x of the equation (E) is called oscillatory if for every $M\in\mathbb{N}$ there exist $m,n\in\mathbb{N}$, $M\le m<n$ such that $x_mx_n<0$, otherwise, it is called nonoscillatory. In other words, a solution x is called nonoscillatory if it is eventually positive or eventually negative. Every solution of (E) is nonoscillatory. If $x=\{x_n\}$ is a solution of (E), then $-x=\{-x_n\}$ is also a solution. Why? (Answer) Thus, there is no need to investigate both cases, and we restrict ourselves to solutions that are eventually positive. Any such solution $\{x_n\}$ is eventually strongly monotone and belongs to one of the two classes

$$\begin{aligned}
\mathbb{M}^+ &= \{x \text{ solution of } (E)\,|\,\exists\, n_0\ge 1\ :\ x_n>0,\quad \Delta x_n>0,\quad \text{for } n\ge n_0\}\,,\\
\mathbb{M}^- &= \{x \text{ solution of } (E)\,|\,x_n>0,\quad \Delta x_n<0,\quad \text{for } n\ge 1\}\,.
\end{aligned}$$

The first class is a set of solutions of (E) that are eventually positive and increasing, and the second class represents positive and decreasing solutions of equation (E). In previous classes, we dealt with differential equation like this one (1.1) and we showed that this equation can have an extinct singular solution, i.e. a solution that for all $t>T_x$, where T_x is some real number, is equal to zero. On the contrary, such solutions of the difference equation (E) do not exist. What's really interesting here is one more difference between differential and difference equations. Classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be empty for the differential equation (1.1), but not for the difference equation (E). If you want to find out more about it, check the reference given in the handout. Now, I'd like to introduce you the term "quasi-difference". It's expression that appears in the bracket on the left hand side of our equation, i.e. $x_n^{[1]}=p_n|\Delta x_n|^{\alpha-1}\Delta x_n$ and $x^{[1]}=\{x_n^{[1]}\}$. We divide the classes $\mathbb{M}^+$ and $\mathbb{M}^-$ into the following subclasses:

$$\begin{aligned}
\mathbb{M}^+_{\infty,\infty} &= \{x\in\mathbb{M}^+ : \lim_n x_n=\infty,\quad \lim_n x_n^{[1]}=\infty,\}\,,\\
\mathbb{M}^+_{\infty,l} &= \{x\in\mathbb{M}^+ : \lim_n x_n=\infty,\quad \lim_n x_n^{[1]}=l,\ 0<l<\infty\}\,,\\
\mathbb{M}^+_{k,\infty} &= \{x\in\mathbb{M}^+ : \lim_n x_n=k,\quad 0<k<\infty,\quad \lim_n x_n^{[1]}=\infty\}\,,\\
\mathbb{M}^+_{k,l} &= \{x\in\mathbb{M}^+ : \lim_n x_n=k,\quad 0<k<\infty,\quad \lim_n x_n^{[1]}=l,\ 0<l<\infty\}\,,\\
\mathbb{M}^-_{k,l} &= \{x\in\mathbb{M}^- : \lim_n x_n=k,\quad 0<k<\infty,\quad \lim_n x_n^{[1]}=-l,\ 0<l<\infty\}\,,\\
\mathbb{M}^-_{0,l} &= \{x\in\mathbb{M}^- : \lim_n x_n=0,\quad \lim_n x_n^{[1]}=-l,\quad 0<l<\infty\}\\
\mathbb{M}^-_{k,0} &= \{x\in\mathbb{M}^- : \lim_n x_n=k,\quad 0<k<\infty,\quad \lim_n x_n^{[1]}=0\}\,,\\
\mathbb{M}^-_{0,0} &= \{x\in\mathbb{M}^- : \lim_n x_n=0,\quad \lim_n x_n^{[1]}=0\}\,.
\end{aligned}$$

A *strongly increasing* solution is one from $\mathbb{M}^+_{\infty,\infty}$ and *strongly decreasing* solutions are solutions from class $\mathbb{M}^-_{0,0}$. For solutions which tends to some constant we use $\mathbb{M}^-_B=\mathbb{M}^-_{k,0}\cup\mathbb{M}^-_{k,l}$, $\mathbb{M}^+_B=\mathbb{M}^+_{k,\infty}\cup\mathbb{M}^+_{k,l}$ and for decreasing solutions which tends to zero we use $\mathbb{M}^-_0=\mathbb{M}^-_{0,l}\cup\mathbb{M}^-_{0,0}$. The existence of the solutions of an equation (E) will depend on whether the sum

$$S=\sum_{n=1}^{\infty}\frac{1}{p_n^{1/\alpha}}$$

(ii) Equation (E) has solutions in $\mathbb{M}^-_{0,l}$ *if and only if*

$$J = \sum_{n=1}^{\infty} q_n \left(\sum_{k=n}^{\infty} \frac{1}{p_{k+1}^{1/\alpha}} \right)^{\beta} < \infty.$$

The assertion *(i)* follows from [6, Theorem 2 and Theorem 5-(a)], while the assertion *(ii)* follows from [8, Theorem 2.2 and Theorem 3.1] and [27, Theorem 9].

As regards to the existence of strongly decreasing solutions, it is an open problem. The existence of strongly decreasing solutions in the continuous case, that is for the differential equation (1.1), can be proved as in [37] with the help of fixed point theory by proving that the operator

$$(\mathcal{F}x)(t) = \int_t^{\infty} \left(\frac{1}{p(s)} \int_s^{\infty} q(r) x(r)^{\beta}\, dr \right)^{\frac{1}{\alpha}} ds$$

has a nonzero fixed point. To this end the operator $\mathcal{F}$ acts on the set

$$\Omega = \{x \in C[t_0, \infty] : z(t) \le x(t) \le z(t_0),\ t \ge t_0\},$$

where z is a singular solution of the first kind of (1.1). The second approach, due to [34], is to construct the sequence $\{x_n\}$ of asymptotically constant solutions of differential equation (1.1), having the limit function x, and it gives rise to a positive strongly decreasing solution of (1.1). This approach, however, requires lower bound for such a sequence of solutions, which is again given by a singular solution of the first kind of (1.1). Clearly, due to the nonexistence of singular solutions in the discrete case, neither of these two approaches work.

The recent development of asymptotic analysis of ordinary differential equations by means of regularly varying functions (see [17]-[19],[23]-[26], [29], [33]-[35] and monograph [28] for results up to 2000.), suggests to investigate the discrete problem of the existence of strongly decreasing solutions in the framework of regularly varying sequences. The aim of this paper is twofold. We will determine conditions for the existence of strongly decreasing solutions and give an explicit asymptotic formula for those solutions.

The theory of regularly varying sequences, sometimes called Karamata sequences, was initiated in 1930 by Karamata [22] and further developed in the seventies by Galambos, Seneta and Bojanić in [5, 16] and recently in [14, 15]. However, until the papers of Matucci and Rehak [30, 31], the relation between regularly varying sequences and difference equations has never been discussed. In these two papers, as well as in succeeding papers [32, 36], the theory of regularly varying sequences has been further developed and applied in the asymptotic analysis of second-order linear and half-linear difference equations, providing necessary and sufficient conditions for the existence of regularly varying solutions of these equations. Afterward, further development of discrete regularly varying theory and application to second-order nonlinear difference equations of Emden-Fowler type was done by Agarwal and Manojlović in [3], Kapešić and Manojlović in [21] and Kapešić in [20]. Actually, in [21], Kapešić and Manojlović gave necessary and sufficient conditions for the existence of strongly increasing regularly varying solutions of (E) and obtained a precise asymptotic representation of such solutions. Thus, the purpose of this paper is to proceed further in this direction and to establish results which can be considered as a discrete analog of results in the continuous case (see e.g.[17, 25, 29]).

Throughout this paper, symbol ~ is used to denote the asymptotic equivalence of two positive sequences, i.e.

$$x_n \sim y_n,\ n \to \infty \iff \lim_{n\to\infty} \frac{y_n}{x_n} = 1.$$

Our main tools are, besides the theory of regularly varying sequences presented in Section 2, the fixed point technique and Stolz-Cezaro theorem. Thus, we recall two variants of Stolz-Cezaro theorem as well as Knaster-Tarski fixed point theorem [1, Theorem 5.2.1].

is convergent or divergent. Really, if S is divergent, then our equation does not have increasing solutions that tend to constant as well as decreasing solutions whose quasi-difference tends to a negative constant. When S is convergent, we know that there cannot be increasing solutions that tend to infinity and whose quasi-difference tends to a positive constant.

(i) if $S = \infty$ then

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_{\infty,l} \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_{k,0} \cup \mathbb{M}^-_{0,0}, \quad \text{i.e.} \quad \mathbb{M}^+_B = \emptyset,\ \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_{k,l} = \emptyset.$$

(ii) If $S < \infty$ then

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_B \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_0 \cup \mathbb{M}^-_B, \quad \text{i.e.} \quad \mathbb{M}^+_{\infty,l} = \emptyset.$$

Because we're talking about decreasing solutions, we're only interested in solutions from the $\mathbb{M}^-$.
Concerning the existence of solutions in the classes $\mathbb{M}^-_B$ and $\mathbb{M}^-_{0,l}$, the following holds.

Theorem 1.1 *(i) Equation (E) has solutions in* $\mathbb{M}^-_B$ *if and only if*

$$I = \sum_{n=1}^{\infty} \left(\frac{1}{p_n} \sum_{k=n}^{\infty} q_k \right)^{\frac{1}{\alpha}} < \infty.$$

(ii) Equation (E) has solutions in $\mathbb{M}^-_{0,l}$ *if and only if*

$$J = \sum_{n=1}^{\infty} q_n \left(\sum_{k=n}^{\infty} \frac{1}{p_{k+1}^{1/\alpha}} \right)^{\beta} < \infty.$$

Let's go back to the differential equation. As we saw, there are two approaches how to determine conditions for the existence of solutions. On the other hand, in the discrete case, neither of these two approaches works. Therefore, we will use the theory of regular variation in order to solve our problem and find asymptotic formulas for solutions of equation (E). Is there anyone who wants to tell us the definition of regularly varying sequences? Which is the most famous theorem? Our main tools are, besides the theory of regularly varying sequences, the fixed point technique and Stolz-Cezaro theorem. Which fixed point theorem we often use? What is the Stolz-Cezaro theorem about?
After this preliminary step, we can go back to the main procedure. We assume that p and q are regularly varying sequences of indices η and σ respectively. How can we express them? Yes, we use the following expressions:

(1.2) $$p_n = n^{\eta} \xi_n \quad q_n = n^{\sigma} \omega_n, \quad \xi = \{\xi_n\},\ \omega = \{\omega_n\} \in \mathcal{SV}.$$

Since, we are looking for strongly decreasing $\mathcal{RV}-$solutions we will expressed them in the same way

(1.3) $$x_n = n^{\rho} l_n, \quad l = \{l_n\} \in \mathcal{SV}.$$

Because of the computational difficulty, we do not consider the case $\eta = \alpha$. Which cases remain to be investigated? Yes, we distinguish two cases, $\eta < \alpha$ and $\eta > \alpha$. The first

Lemma 1.2. *If $f = \{f_n\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f_n = \infty$, then for any sequence $g = \{g_n\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty} \frac{\Delta f_n}{\Delta g_n} \le \liminf_{n\to\infty} \frac{f_n}{g_n} \le \limsup_{n\to\infty} \frac{f_n}{g_n} \le \limsup_{n\to\infty} \frac{\Delta f_n}{\Delta g_n}.$$

In particular, if the sequence $\{\Delta f_n/\Delta g_n\}$ has a limit, then

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \lim_{n\to\infty} \frac{\Delta f_n}{\Delta g_n}. \tag{1.2}$$

Lemma 1.3. *Let $f = \{f_n\}$, $g = \{g_n\}$ be sequences of positive real numbers, such that*
(i) $\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n = 0$;
(ii) the sequence g is strictly monotone;
(iii) the sequence $\{\Delta f_n/\Delta g_n\}$ has a limit.
Then, a sequence $\{f_n/g_n\}$ is convergent and (1.2) *holds.*

Lemma 1.4. (Knaster-Tarski fixed point theorem) *Let X be a partially ordered Banach space with ordering $\le$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $\mathcal{F} : M \to M$ be an increasing mapping, i.e. $x \ge y$ implies $\mathcal{F}x \ge \mathcal{F}y$. Then $\mathcal{F}$ has a fixed point in M.*

2. Regularly Varying Sequences

We state here definitions and some basic properties of regularly varying sequences which will be essential in establishing our main results on the asymptotic behavior of nonoscillatory solutions stated and proved in the next section. For a comprehensive treatise on regular variation, the reader is referred to Bingham et al. [4].

Two main approaches are known in the basic theory of regularly varying sequences: the approach due to Karamata [22], based on a definition that can be understood as a direct discrete counterpart of elegant and straightforward continuous definition (see Definition 2.2), and the approach due to Galambos and Seneta, based on purely sequential definition.

Definition 2.1. (Karamata [22]) A positive sequence $y = \{y_k\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y_{[\lambda k]}}{y_k} = \lambda^\rho \quad \text{for} \quad \forall \lambda > 0,$$

where $[u]$ denotes the integer part of u.

Definition 2.2. A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be *regularly varying at infinity of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all} \ \lambda > 0.$$

Definition 2.3. (Galambos and Seneta [16]) A positive sequence $y = \{y_k\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha_k\}$ satisfying

$$\lim_{k\to\infty} \frac{y_k}{\alpha_k} = C,\ 0 < C < \infty \qquad \lim_{k\to\infty} k\frac{\Delta \alpha_{k-1}}{\alpha_k} = \rho.$$

If $\rho = 0$, then y is said to be *slowly varying*.

implies that S is divergent, and the second implies that S is convergent. In the first case, any strongly decreasing solution of (E) is less than or equal to some constant. What can we say about the regularity index? If $\rho = 0$ is x is a trivial or non trivial RV-solution?
If $\eta > \alpha$, using discrete Karamata's theorem, we have

$$\pi_n = \sum_{k=n}^{\infty} \frac{1}{p_k^{1/\alpha}} = \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \xi_k^{-\frac{1}{\alpha}} \sim \frac{\alpha}{\eta-\alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} = \frac{\alpha}{\eta-\alpha} \cdot \frac{n}{p_n^{1/\alpha}}, \quad n \to \infty, \tag{1.4}$$

so that $\{\pi_n\} \in \mathcal{RV}(\frac{\alpha-\eta}{\alpha})$. Since,

$$\lim_{n\to\infty} \frac{x_n}{\pi_n} = \lim_{n\to\infty} \frac{\Delta x_n}{-\frac{1}{p_n^{\frac{1}{\alpha}}}} = \lim_{n\to\infty} (x_n^{[1]})^{\frac{1}{\alpha}} = 0,$$

we conclude that index of regularity strongly decreasing solutions is less then or equal to index of regularity of π_n, which is $\frac{\alpha-\eta}{\alpha}$. What happens if $\rho = \frac{\alpha-\eta}{\alpha}$? Whether these solutions are trivial or non-trivial?
We already indicate that continuous and discrete case have differences, but they also have similarities. Does anyone remember under what conditions the differential equation (1.1) has strongly decreasing solutions? What I want to emphasize is that we will obtain the similar result here. Actually, we have

Theorem 1.2 *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. (i) Let $\eta < \alpha$. If $I < \infty$, then $\mathbb{M}^-_{0,0} \neq \emptyset$. (ii) Let $\eta > \alpha$. If $J < \infty$, then $\mathbb{M}^-_{0,0} \neq \emptyset$.*

Series I and J are given in the first part of the lecture. So, to clarify, our equation has a strongly decreasing solution if I in the first case and J in the second are convergent. In order to prove the previous theorem, we can see that (i) for $\eta < \alpha$, $I < \infty$ if and only if

$$\sigma < \eta - \alpha - 1 \tag{1.5}$$

or

$$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1} \left(\frac{\omega_k}{\xi_k}\right)^{\frac{1}{\alpha}} < \infty; \tag{1.6}$$

(ii) for $\eta > \alpha$, $J < \infty$ if and only if

$$\sigma < \frac{\beta\eta}{\alpha} - \beta - 1 \tag{1.7}$$

or

$$\sigma = \frac{\beta\eta}{\alpha} - \beta - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1} \frac{\omega_k}{\xi_k^{\beta/\alpha}} < \infty. \tag{1.8}$$

Taking the preceding considerations into account, the Theorem 1.2 will be proven by considering these four cases.

The totality of regularly varying sequences of index ρ and slowly varying sequences will be denoted, respectively, by $\mathcal{RV}(\rho)$ and $\mathcal{SV}$.

Bojanić and Seneta have shown in [5] that Definition 2.1 and Definition 2.3 are equivalent.

The concept of normalized regularly varying sequences was introduced by Matucci and Rehak in [30]:

Definition 2.4. A positive sequence $y = \{y_k\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y_k}{y_k} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

In what follows, $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ will be used to denote the set of all normalized regularly varying sequences of index ρ and the set of all normalized slowly varying sequences. Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^\rho \log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}.$$

There exist various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying (see [5, 16, 30, 31]), and consequently, each one of them could be used to define a regularly varying sequence. The one that is the most important is the following Representation theorem (see [5, Theorem 3]), while some other representation formula for regularly varying sequences were established in [31, Lemma 1].

Theorem 2.5. (Representation theorem) *A positive sequence $\{y_k\}, k \in \mathbb{N}$ is said to be regularly varying of index $\rho \in \mathbb{R}$ if and only if there exist sequences $\{c_k\}$ and $\{\delta_k\}$ such that*

$$\lim_{k\to\infty} c_k = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{k\to\infty} \delta_k = 0,$$

and

$$y_k = c_k k^\rho \exp\left(\sum_{i=1}^{k} \frac{\delta_i}{i}\right).$$

In [5] very useful embedding theorem was proved, which gives the possibility of using the continuous theory in developing a theory of regularly varying sequences. However, as noticed in [5], such development is not generally close and sometimes far from a simple imitation of arguments for regularly varying functions.

Theorem 2.6. (Embedding Theorem) *If $y = \{y_n\}$ is a regularly varying sequence of index $\rho \in \mathbb{R}$, then function $Y(t)$ defined on $[0,\infty)$ by $Y(t) = y_{[t]}$ is a regularly varying function of index ρ. Conversely, if $Y(t)$ is a regularly varying function on $[0,\infty)$ of index ρ, then a sequence $\{y_k\}$, $y_k = Y(k)$, $k \in \mathbb{N}$ is regularly varying of index ρ.*

Next, we state some important properties of $\mathcal{RV}$ sequences, useful for the development of asymptotic behavior of solutions of (E) in the subsequent section (for more properties and proofs see [5, 30]).

Theorem 2.7. *(i) $y \in \mathcal{RV}(\rho)$ if and only if $y_k = k^\rho l_k$, where $l = \{l_k\} \in \mathcal{SV}$.*

(ii) Let $x \in \mathcal{RV}(\rho_1)$ and $y \in \mathcal{RV}(\rho_2)$. Then, $xy \in \mathcal{RV}(\rho_1 + \rho_2)$, $x + y \in \mathcal{RV}(\rho)$, $\rho = \max\{\rho_1, \rho_2\}$ and $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) If $y \in \mathcal{RV}(\rho)$, then $\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = 1$.

(iv) If $l \in \mathcal{SV}$ and $l_k \sim L_k$, $k \to \infty$, then $L \in \mathcal{SV}$.

(v) If $y \in \mathcal{RV}(\rho)$, then $\{n^{-\sigma} y_n\}$ is eventually increasing for each $\sigma < \rho$ and $\{n^{-\mu} y_n\}$ is eventually decreasing for each $\mu > \rho$.

Theorem 1.3 *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$. (i) Let $\eta < \alpha$. If (1.5) holds, then equation (E) possesses a solution $x \in \mathbb{M}^-_{0,0}$. (ii) Let $\eta > \alpha$. If (1.7) holds, then equation (E) possesses a solution $x \in \mathbb{M}^-_{0,0}$.*

In both cases, we can easily prove that the sequence $X = \{X_n\}$,

$$X_n = \left[\frac{n^{\alpha+1} p_n^{-1} q_n}{(-\rho)^\alpha (\alpha - \eta - \rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \geq 1, \tag{1.9}$$

where ρ is given by

$$\rho = \frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta} \tag{1.10}$$

satisfy asymptotic relation

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^\beta\right)^{\frac{1}{\alpha}} \sim X_n, \qquad n \to \infty. \tag{1.11}$$

Also, if we look at how we defined X_n, we can conclude that its index of regularity is ρ and that $X_n \to \infty$. Thus, there exists $n_0 > 1$ such that

$$X_{n+1} \leq X_n \quad \text{and} \quad \frac{1}{2} X_n \leq \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^\beta\right)^{\frac{1}{\alpha}} \leq 2X_n, \quad \text{for } n \geq n_0\,. \tag{1.12}$$

Let such n_0 be fixed. We choose constants $\kappa \in (0,1)$ and $K > 1$ such that

$$\kappa^{1-\frac{\beta}{\alpha}} \leq \frac{1}{2} \qquad \text{and} \qquad K^{1-\frac{\beta}{\alpha}} \geq 2\,. \tag{1.13}$$

Consider the space Υ_{n_0} of all real sequences $x = \{x_n\}_{n=n_0}^\infty$ such that x_n/X_n is bounded for $n \geq n_0$. Then, Υ_{n_0} is a Banach space, endowed with the norm

$$||x|| = \sup_{n\geq n_0} \frac{x_n}{X_n}\,.$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\leq$: for $x, y \in \Upsilon_{n_0}$, $x \leq y$ means $x_n \leq y_n$ for all $n \geq n_0$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ by

$$\mathcal{X} = \{x \in \Upsilon_{n_0} : \kappa X_n \leq x_n \leq K X_n,\, n \geq n_0\}. \tag{1.14}$$

For any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. Next, define the operator $\mathcal{F} : \mathcal{X} \to \Upsilon_{n_0}$ by

$$(\mathcal{F}x)_n = \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j x_{j+1}^\beta\right)^{\frac{1}{\alpha}}, \quad n \geq n_0, \tag{1.15}$$

and show that $\mathcal{F}$ has a fixed point. Which theorem will we use? What are the properties of the Knaster-Tarski fixed point theorem? Are all of them fulfilled? OK, $\mathcal{F}$ has fixed point x. That fixed point is a solution that we are looking for. Why? What do we need to check? Similarly, if $\eta < \alpha$ and (1.6) hold or $\eta > \alpha$ and (1.8) hold, then equation (E) has a strongly decreasing solution.

In the next part of the lecture, we discuss the asymptotic representation of strongly decreasing solutions.

(vi) *Let* $l \in \mathcal{SV}$. *Then,* $\lim_{n\to\infty} n^{\rho}\, l_n = 0$ *if* $\rho < 0$ *and* $\lim_{n\to\infty} n^{\rho}\, l_n = \infty$ *if* $\rho > 0$.

In view of the statement (*i*) of the previous theorem, if for $y \in \mathcal{RV}(\rho)$

$$\lim_{k\to\infty} \frac{y_k}{k^{\rho}} = \lim_{k\to\infty} l_k = \text{const} > 0,$$

then $y = \{y_n\}$ is said to be a *trivial regularly varying sequence of index* ρ and is denoted by $y \in tr-\mathcal{RV}(\rho)$. Otherwise y is said to be a *nontrivial regularly varying sequence of index* ρ, denoted by $y \in ntr-\mathcal{RV}(\rho)$.

Next theorem can be found in [3] for normalized regularly varying sequences, but it apparently holds for all regularly varying sequences.

Theorem 2.8. *If* $f = \{f_n\} \in \mathcal{RV}$ *is a strictly decreasing sequence, such that* $\lim_{n\to\infty} f_n = 0$, *then for each* $\gamma \in \mathbb{R}$

$$\lim_{n\to\infty} f_n^{-\gamma} \sum_{k=n}^{\infty} f_k^{\gamma-1}\left(-\Delta f_k\right) = \frac{1}{\gamma}.$$

If $g = \{g_n\} \in \mathcal{RV}$ *is a strictly increasing sequence such that* $\lim_{n\to\infty} g_n = \infty$, *then*

$$\lim_{n\to\infty} g_n^{-\gamma} \sum_{k=1}^{n-1} g_k^{\gamma-1} \Delta g_k = \frac{1}{\gamma}.$$

The following theorem can be seen as *the discrete analog of the Karamata's integration theorem* and plays a central role in the proof of our main results in Section 3. The proof of this theorem can be found in [5], [21] and [36].

Theorem 2.9. *Let* $l = \{l_n\} \in \mathcal{SV}$.

(*i*) *If* $\alpha > -1$, *then* $\displaystyle\lim_{n\to\infty} \frac{1}{n^{\alpha+1} l_n} \sum_{k=1}^{n} k^{\alpha} l_k = \frac{1}{1+\alpha}$;

(*ii*) *If* $\alpha < -1$, *then* $\displaystyle\lim_{n\to\infty} \frac{1}{n^{\alpha+1} l_n} \sum_{k=n}^{\infty} k^{\alpha} l_k = -\frac{1}{1+\alpha}$;

(*iii*) *If* $\displaystyle\sum_{k=1}^{\infty} \frac{l_k}{k} < \infty$, *then* $\displaystyle\sum_{k=n}^{\infty} \frac{l_k}{k} \in \mathcal{SV}$ *and* $\displaystyle\lim_{n\to\infty} \frac{1}{l_n} \sum_{k=n}^{\infty} \frac{l_k}{k} = \infty$;

(*iv*) *If* $\displaystyle\sum_{k=1}^{\infty} \frac{l_k}{k} = \infty$, *then* $\displaystyle\sum_{k=1}^{n} \frac{l_k}{k} \in \mathcal{SV}$ *and* $\displaystyle\lim_{n\to\infty} \frac{1}{l_n} \sum_{k=1}^{n} \frac{l_k}{k} = \infty$.

Remark 2.10. In view of Theorem 2.7-(iii) and Theorem 2.9-(i), it is easy to see, that for $l \in \mathcal{SV}$, if $\alpha > -1$, then we have

$$\sum_{k=1}^{n-1} k^{\alpha} l_k \sim \frac{(n-1)^{\alpha+1} l_{n-1}}{\alpha+1} \sim \frac{n^{\alpha+1} l_n}{\alpha+1} \sim \sum_{k=1}^{n} k^{\alpha} l_k, \qquad n\to\infty.$$

3. Main results

In this section we assume that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and use expressions

$$p_n = n^{\eta}\xi_n \quad q_n = n^{\sigma}\omega_n, \quad \xi = \{\xi_n\},\ \omega = \{\omega_n\} \in \mathcal{SV}\,, \tag{3.1}$$

considering strongly decreasing $\mathcal{RV}$−solutions expressed as

$$x_n = n^{\rho} l_n, \quad l = \{l_n\} \in \mathcal{SV}\,. \tag{3.2}$$

Theorem 1.4 *Suppose that* $p \in \mathcal{RV}(\eta)$ *and* $q \in \mathcal{RV}(\sigma)$. (i) *Let* $\eta < \alpha$. *Equation* (E) *possesses regularly varying solutions* x *of index* $\rho < 0$ *if and only if* (1.5) *holds.* (ii) *Let* $\eta > \alpha$. *Equation* (E) *possesses regularly varying solutions* x *of index* $\rho < \frac{\alpha-\eta}{\alpha}$ *if and only if* (1.7). *In both cases* ρ *is given by* (1.10) *and the asymptotic behavior of any such solution* x *is governed by the unique formula* (1.16).

Proof. **The "only if" part:** If $\eta < \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho < 0$ then x is strongly decreasing. Why? Summing (E) twice from n to ∞ and using discrete Karamata's theorem, we obtain that x has asymptotic behavior

$$x_n \sim \left[\frac{n^{\alpha+1} p_n^{-1} q_n}{(-\rho)^{\alpha}(\alpha-\eta-\rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \quad n\to\infty. \tag{1.16}$$

Also, the index of regularity is given by (1.10) and σ satisfies (1.5). In the opposite case, for $\eta > \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho < \frac{\alpha-\eta}{\alpha}$, we see that (1.7) holds and that ρ is given by (1.10). **The "if" part:** According to the previous theorem, equation (E) has a solution $x \in \mathbb{M}^{-}_{0,0}$. It remains to prove that x is a regularly varying sequence of index ρ. Do you remember how we showed this in a continuous case? Here we use the Stolz-Cezaro theorem. The procedure is the same as in continuous case. We have

$$0 < \liminf_{n\to\infty} \frac{x_n}{X_n} \le \limsup_{n\to\infty} \frac{x_n}{X_n} < \infty,$$

where X_n is given by (1.9). Then,

$$\begin{aligned}
L &= \limsup_{n\to\infty} \frac{x_n}{X_n} \le \limsup_{n\to\infty} \frac{\Delta x_n}{\Delta X_n} = \limsup_{n\to\infty} \frac{-\left(\frac{1}{p_k}\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}\right)^{1/\alpha}}{-\left(\frac{1}{p_k}\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta}\right)^{1/\alpha}} \\
&\le \left(\limsup_{n\to\infty} \frac{\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}}{\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta}}\right)^{1/\alpha} \le \left(\limsup_{n\to\infty} \frac{-q_n x_{n+1}^{\beta}}{-q_n X_{n+1}^{\beta}}\right)^{1/\alpha} \\
&\le \left(\limsup_{n\to\infty} \frac{x_{n+1}}{X_{n+1}}\right)^{\beta/\alpha} = L^{\frac{\beta}{\alpha}}.
\end{aligned}$$

Since $\beta < \alpha$, we conclude that

$$0 < L \le 1\,. \tag{1.17}$$

Similarly, we can see that $l = \liminf_{n\to\infty} x_n/X_n$ satisfies

$$1 \le l < \infty\,. \tag{1.18}$$

From (1.17) and (1.18) we obtain that $l = L = 1$, which means that $x_n \sim X_n$, $n\to\infty$ and ensures that x is a regularly varying solution of (E) with requested regularity index and the asymptotic representation given by (1.16). □We consider border cases separately.

Theorem 1.5 *Suppose that* $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ *and* $q \in \mathcal{RV}(\sigma)$. *There exists* $x \in \mathbb{M}^{-}_{0,0} \cap ntr-\mathcal{SV}$ *if and only if* (1.6) *holds. All such solutions of* (E) *enjoy the precise asymptotic formula*

$$x_n \sim \left[\frac{\alpha-\beta}{\alpha} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\alpha-\beta}}, \quad n\to\infty. \tag{1.19}$$

Moreover, we assume that $\eta \neq \alpha$ and distinguish two mutually exclusive cases:

(i) $\eta < \alpha$ implying that $S = \infty$; and (ii) $\eta > \alpha$ implying that $S < \infty$.

Case (i): It is clear that for any strongly decreasing solution of (E) it holds that $x_n \leq c$, for large n. Thus, we have that the index of regularity ρ of strongly decreasing $\mathcal{RV}-$solution x must satisfy $\rho \leq 0$. If $\rho = 0$ then $l_n = x_n \to 0$, so x is a member of $ntr - \mathcal{SV}$.

Case (ii): Using (3.1) and Theorem 2.9 we have

$$\pi_n = \sum_{k=n}^{\infty} \frac{1}{p_k^{1/\alpha}} = \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \xi_k^{-\frac{1}{\alpha}} \sim \frac{\alpha}{\eta - \alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} = \frac{\alpha}{\eta - \alpha} \cdot \frac{n}{p_n^{1/\alpha}}, \quad n \to \infty, \tag{3.3}$$

so that $\{\pi_n\} \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. For any strongly decreasing solution x of (E), by application of Lemma 1.3, we have that

$$\lim_{n\to\infty} \frac{x_n}{\pi_n} = \lim_{n\to\infty} \frac{\Delta x_n}{-\frac{1}{p_n^{\frac{1}{\alpha}}}} = \lim_{n\to\infty} (x_n{}^{[1]})^{\frac{1}{\alpha}} = 0,$$

implying that the index of regularity ρ of strongly decreasing solutions must satisfy $\rho \leq \frac{\alpha-\eta}{\alpha}$.

If $\eta < \alpha$, the totality of strongly decreasing $\mathcal{RV}-$solutions will be divided into the following two classes

$$ntr - \mathcal{SV} \quad \text{or} \quad \mathcal{RV}(\rho) \text{ with } \rho < 0,$$

while, if $\eta > \alpha$, the totality of strongly decreasing $\mathcal{RV}-$solutions of (E) will be divided into the following two subclasses:

$$\mathcal{RV}\left(\frac{\alpha - \eta}{\alpha}\right) \quad \text{or} \quad \mathcal{RV}(\rho) \text{ with } \rho < \frac{\alpha - \eta}{\alpha}.$$

Our purpose is to show that all solutions in each of this four subclasses of strongly decreasing $\mathcal{RV}-$solutions of (E) enjoy one and the same asymptotic behavior as $n \to \infty$, whereby the regularity index of such a solution is uniquely determined by α, β and the regularity indices η, σ of coefficients p, q. Moreover, necessary and sufficient conditions for the existence of solutions belonging to these four subclasses of strongly decreasing $\mathcal{RV}-$solutions will be established.

3.1. *Existence of strongly decreasing solutions*

Conditions for the existence of a strongly decreasing solution of differential equation (1.1) is given by the following theorem:

Theorem 3.1. (i) *Let* $\int_a^\infty p(t)^{-\frac{1}{\alpha}}\, dt = \infty$, $a \geq 0$. *If*

$$\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty q(s) ds \right)^{\frac{1}{\alpha}} dt < \infty,$$

then equation (1.1) *has a strongly decreasing solution.*

(ii) *Let* $\int_a^\infty p(t)^{-\frac{1}{\alpha}}\, dt < \infty$, $a \geq 0$. *If*

$$\int_a^\infty q(t) \left(\int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}} \right)^{\beta} dt < \infty,$$

then equation (1.1) *has a strongly decreasing solution.*

Proof. **The "only if" part:** We can calculate the index of regularity of solution x. Could anyone tell me what ρ is? Similarly, as in the previous theorem, we obtain that x i given with

$$x_n \sim \frac{1}{(\alpha - \eta)^{\frac{1}{\alpha}}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{1.20}$$

By properties of RVsequences and discrete Karamata's theorem, we prove that x has the asymptotic formula (1.19) and that (1.6) holds. **The "if" part:** In the same way as in the previous theorem, we get that equation (E) has a strongly decreasing RVsolution. □Another border case is given by the next theorem.

Theorem 1.6 *Suppose that* $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ *and* $q \in \mathcal{RV}(\sigma)$. *There exists* $x \in \mathbb{M}_{0,0}^- \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$ *if and only if* (1.8) *holds. All such solutions of (E) enjoy the precise asymptotic behaviour*

$$x_n \sim \left(\alpha^{\alpha-1} \frac{\alpha - \beta}{(\eta - \alpha)^{\alpha}} \right)^{\frac{1}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}} \left[\sum_{k=n}^{\infty} k^{\beta} q_k p_k^{-\frac{\beta}{\alpha}} \right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{1.21}$$

I do not want to prove this theorem now since its proof is quite similar to the proof of the previous theorem. For more information, you can read the literature given in the hangout. Now, when we have proved all the results, what can we conclude? I'd like to emphasize that the existence of strongly decreasing $\mathcal{RV}-$solutions for the equation (E) with $\mathcal{RV}$ coefficients is fully characterized by the assumption $I < \infty$ if $S = \infty$ and by the assumption $J < \infty$ if $S < \infty$. In fact, this conclusion can be formulated in the following way.

Corollary 1.1 *Suppose that* $p \in \mathcal{RV}(\eta)$, $\eta \neq \alpha$ *and* $q \in \mathcal{RV}(\sigma)$.

(i) Let $S = \infty$. *Equation (E) has strongly decreasing* $\mathcal{RV}$*-solutions if and only if* $I < \infty$.

(ii) Let $S < \infty$. *Equation (E) has strongly decreasing* $\mathcal{RV}$*-solutions if and only if* $J < \infty$.

Moreover, if $S = \infty$, then $J = \infty$ so by Theorem 1.1 $\mathbb{M}_{0,l}^- = \emptyset$. Otherwise, if $S < \infty$, denoting the series $Q = \sum_{k=1}^{\infty} q_k$, we have two cases:

(a) If $Q = \infty$, then $I = \infty$, so by Theorem 1.1 we have $\mathbb{M}^- = \mathbb{M}_0^-$ i.e. $\mathbb{M}_B^- = \emptyset$.

(b) If $Q < \infty$, then $I < \infty$, so by Theorem 1.1 we have $\mathbb{M}^- = \mathbb{M}_0^- \cup \mathbb{M}_B^-$.

Using conclusions from the previous corollary and Theorem 1.1, we get the next two corollaries where we will use the following symbols: $*\ \mathcal{R}$ denote the set of all regularly varying solutions, $*\ \mathcal{R}^-$ denote the set of all decreasing regularly varying solutions, $*\ \mathcal{R}_0^- = \mathcal{R} \cap \mathbb{M}_0^-$. $*\ \mathcal{R}_{0,0}^- = \mathcal{R} \cap \mathbb{M}_{0,0}^-$.

Corollary 1.2 *Suppose that* $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ *and* $S = \infty$. *Then,*

$$\mathcal{R}^- = ntr - \mathcal{SV} \cup \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta} \right) \cup \mathbb{M}_B^-$$

if and only if $I < \infty$.

The proof of the previous theorem can be found in [13, 34] for (*i*) and in [37] for (*ii*). It is expected that for the discrete version of Theorem 3.1, the existence of strongly decreasing solution is characterized by the assumption $I < \infty$ if $S = \infty$ and by the assumption $J < \infty$ if $S < \infty$. In fact, we prove

Theorem 3.2. *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$.*
(i) Let $\eta < \alpha$. If $I < \infty$, then $\mathbb{M}^-_{0,0} \neq \emptyset$.
(ii) Let $\eta > \alpha$. If $J < \infty$, then $\mathbb{M}^-_{0,0} \neq \emptyset$.

First of all, let's notice that if $\eta < \alpha$, then $\sigma < -1$ is a necessary condition for $I < \infty$. Then, using discrete Karamata theorem, (3.1) and (3.3), we have

$$\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}} \sim \frac{1}{(-(\sigma+1))^{\frac{1}{\alpha}}}\left(\frac{k^{\sigma+1-\eta}\omega_k}{\xi_k}\right)^{\frac{1}{\alpha}}, \quad k \to \infty .$$

On the other hand, if $\eta > \alpha$ application of discrete Karamata theorem gives

$$q_k\left(\sum_{j=k}^{\infty}\frac{1}{p_j^{1/\alpha}}\right)^{\beta} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} k^{\sigma+\beta-\frac{\beta}{\alpha}\eta}\,\frac{\omega_k}{\xi_k^{\beta/\alpha}}, \quad k \to \infty.$$

Consequently,

(i) for $\eta < \alpha$, $I < \infty$ if and only if

$$\sigma < \eta - \alpha - 1 \tag{3.4}$$

or

$$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1}\left(\frac{\omega_k}{\xi_k}\right)^{\frac{1}{\alpha}} < \infty ; \tag{3.5}$$

(ii) for $\eta > \alpha$, $J < \infty$ if and only if

$$\sigma < \frac{\beta\eta}{\alpha} - \beta - 1 \tag{3.6}$$

or

$$\sigma = \frac{\beta\eta}{\alpha} - \beta - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1}\frac{\omega_k}{\xi_k^{\beta/\alpha}} < \infty . \tag{3.7}$$

Taking into account the previous consideration, Theorem 3.2 will be proved by considering the above four cases.

Theorem 3.3. *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$.*
(i) Let $\eta < \alpha$. If (3.4) holds, then equation (E) possesses a solution $x \in \mathbb{M}^-_{0,0}$.
(ii) Let $\eta > \alpha$. If (3.6) holds, then equation (E) possesses a solution $x \in \mathbb{M}^-_{0,0}$.

Proof. Suppose either $\eta < \alpha$ and (3.4) holds or $\eta > \alpha$ and (3.6) holds. Denote

$$X_n = \left[\frac{n^{\alpha+1}p_n^{-1}q_n}{(-\rho)^{\alpha}(\alpha-\eta-\rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \geq 1, \tag{3.8}$$

and $\lambda = (-\rho)^{\alpha}(\alpha - \eta - \rho\alpha)$, where ρ is given by

$$\rho = \frac{\sigma+\alpha+1-\eta}{\alpha-\beta}. \tag{3.9}$$

Corollary 1.3 *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S < \infty$. Then,*

(i) If $\sigma < -1$ or $\sigma = -1$ and $Q < \infty$, then

$$\mathcal{R}^- = \mathcal{RV}\left(\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}\right) \cup \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_B.$$

(ii) If $\sigma = -1$ and $Q = \infty$ or $-1 < \sigma < \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R}^- = \mathcal{R}^-_0 = \mathcal{RV}\left(\frac{\sigma+\alpha+1-\eta}{\alpha-\beta}\right) \cup \mathbb{M}^-_{0,l}.$$

(iii) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J < \infty$, then

$$\mathcal{R}^- = \mathcal{R}^-_0 = \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right) \cup \mathbb{M}^-_{0,l}.$$

(iv) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J = \infty$ or $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R}^- = \emptyset.$$

I want to emphasize one thing that is very interesting. At the beginning of today's lecture, I said that class $\mathbb{M}^-$ is not empty. But, if you look at the previous corollary, you can see that class $\mathcal{R}^- = \emptyset$. Can anyone explain this?
I'd like to illustrate with examples what we learned today.

Example 1.1 Consider the difference equation

$$\Delta\left(\frac{n^{\eta}}{\log n}(\Delta x_n)^3\right) = \frac{n^{\eta-7}\varphi_n}{\log^5 n}\sqrt{x_{n+1}^3}, \quad n \geq 1, \tag{1.22}$$

where φ_n is a positive real-value sequence such that $\lim_{n\to\infty}\varphi_n = \delta$ and $\eta \neq 3$. In this equation, $\alpha = 3$, $\beta = \frac{3}{2}$, $\{p_n\} \in \mathcal{RV}(\eta)$ and $\{q_n\} \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 7$. (i) Suppose that $\eta < 3$. In this case

$$\sigma = \eta - 7 < \eta - 4 = \eta - \alpha - 1,$$

so in view of Theorem 1.4-(i) and Theorem 1.1 this equation has a strongly decreasing $\mathcal{RV}$-solution of index $\rho < 0$ as well as a solution in $\mathbb{M}^-_B = tr - \mathcal{SV}$. More precisely, by Theorem 1.4-(i) equation (1.22) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$. That solution has asymptotic behavior

$$x_n \sim \left(\frac{\delta}{8(9-\eta)}\right)^{\frac{2}{3}} n^{-2}(\log n)^{-\frac{8}{3}}, \quad n \to \infty. \tag{1.23}$$

If

$$\varphi_n = \frac{n^7(n+1)^3}{(\log n)^4(\log(n+1))^5}\left((\log n)^9\psi_n - (\log(n+1))^9\left(\frac{n+1}{n}\right)^{\eta}\psi_{n+1}\right), \tag{1.24}$$

Clearly, $X = \{X_n\} \in \mathcal{RV}(\rho)$ and it may be expressed in the form

$$X_n = \lambda^{-\frac{1}{\alpha-\beta}} n^{\rho} \left(\frac{\omega_n}{\xi_n}\right)^{\frac{1}{\alpha-\beta}} . \tag{3.10}$$

Notice that (3.4) and (3.9) imply that $\rho < 0$, while (3.6) and (3.9) imply that $\rho < \frac{\alpha-\eta}{\alpha}$, so that by Theorem 2.7-(v),(vi), $X_n \to 0$ as $n \to \infty$ and $\{X_n\}$ is eventually decreasing, in both cases (i) and (ii).

Let us first prove that the sequence X satisfies the asymptotic relation

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \sim X_n, \qquad n \to \infty. \tag{3.11}$$

Using (3.1), by application of Theorem 2.9-(ii) and Theorem 2.7-(iii), we get

$$\begin{aligned} \sum_{k=n}^{\infty} q_k X_{k+1}^{\beta} &\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \xi_k^{-\frac{\beta}{\alpha-\beta}} \omega_k^{\frac{\alpha}{\alpha-\beta}} = \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{\alpha(\rho-1)+\eta-1} \xi_k^{-\frac{\beta}{\alpha-\beta}} \omega_k^{\frac{\alpha}{\alpha-\beta}} \\ &\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \frac{n^{\alpha(\rho-1)+\eta} \xi_n^{-\frac{\beta}{\alpha-\beta}} \omega_n^{\frac{\alpha}{\alpha-\beta}}}{-(\alpha(\rho-1)+\eta)}, \quad n \to \infty. \end{aligned} \tag{3.12}$$

Notice that $\alpha(\rho-1)+\eta < 0$ in both cases (i) and (ii). From (3.12), applying Theorem 2.9-(ii), we obtain the desired asymptotic relation for X:

$$\begin{aligned} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} &\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} (\alpha(1-\rho)-\eta)^{-\frac{1}{\alpha}} \sum_{k=n}^{\infty} k^{\rho-1} \xi_k^{-\frac{1}{\alpha-\beta}} \omega_k^{\frac{1}{\alpha-\beta}} \\ &\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} (\alpha(1-\rho)-\eta)^{-\frac{1}{\alpha}} \frac{n^{\rho} \xi_n^{-\frac{1}{\alpha-\beta}} \omega_n^{\frac{1}{\alpha-\beta}}}{-\rho} \\ &= \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \cdot \lambda^{-\frac{1}{\alpha}} n^{\rho} \xi_n^{-\frac{1}{\alpha-\beta}} \omega_n^{\frac{1}{\alpha-\beta}} = X_n, \qquad n \to \infty . \end{aligned}$$

Thus, there exists $n_0 > 1$ such that

$$X_{n+1} \le X_n \quad \text{and} \quad \frac{1}{2} X_n \le \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \le 2X_n, \quad \text{for } n \ge n_0 . \tag{3.13}$$

Let such n_0 be fixed. We choose constants $\kappa \in (0,1)$ and $K > 1$ such that

$$\kappa^{1-\frac{\beta}{\alpha}} \le \frac{1}{2} \qquad \text{and} \qquad K^{1-\frac{\beta}{\alpha}} \ge 2 . \tag{3.14}$$

Consider the space Υ_{n_0} of all real sequences $x = \{x_n\}_{n=n_0}^{\infty}$ such that x_n/X_n is bounded for $n \ge n_0$. Then, Υ_{n_0} is a Banach space, endowed with the norm

$$\|x\| = \sup_{n \ge n_0} \frac{x_n}{X_n} .$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: for $x, y \in \Upsilon_{n_0}$, $x \le y$ means $x_n \le y_n$ for all $n \ge n_0$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ by

$$\mathcal{X} = \{x \in \Upsilon_{n_0} : \kappa X_n \le x_n \le K X_n,\ n \ge n_0\}. \tag{3.15}$$

For any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. Next, define the operator $\mathcal{F} : \mathcal{X} \to \Upsilon_{n_0}$ by

$$(\mathcal{F} x)_n = \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{\frac{1}{\alpha}}, \quad n \ge n_0, \tag{3.16}$$

where

$$\psi_n = \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \left(\frac{\log n}{\log(n+1)} \right)^{\frac{8}{3}} \right)^3 ,$$

then $\delta = 8(9-\eta)$ and the considered equation has an exact solution $n^{-2}(\log n)^{-\frac{8}{3}}$. (ii) For $\eta \in (3,9)$ we have that $\eta > \alpha$ and $\sigma = \eta - 7 < \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$, so in view of Theorem 1.4-(ii) the equation (1.22) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$ and satisfies (1.23). This equation also possess a solution which belongs to a class $\mathbb{M}^-_{0,l}$. (iii) Let $\eta = 9$. Then, $\sigma = 2 = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J < \infty$. By Theorem 1.6 the equation (1.22) has a solution $x \in \mathcal{RV}(1-\frac{\eta}{\alpha}) = \mathcal{RV}(-2)$ and any such solution x has the asymptotic representation

$$\begin{aligned} x_n &\sim \left(\frac{\delta}{16}\right)^{\frac{2}{3}} n^{-2} (\log n)^{\frac{1}{3}} \left(\sum_{k=n}^{\infty} k^{-1} (\log k)^{-\frac{9}{2}} \right)^{\frac{2}{3}} \\ &\sim \left(\frac{\delta}{16}\right)^{\frac{2}{3}} n^{-2} (\log n)^{\frac{1}{3}} \left(\frac{2}{7}\right)^{\frac{2}{3}} (\log n)^{-\frac{7}{3}} = \left(\frac{\delta}{56}\right)^{\frac{2}{3}} (n \log n)^{-2}, \quad n \to \infty, \end{aligned}$$

where we used that

$$\sum_{k=n}^{\infty} k^{-1} (\log k)^{-\frac{9}{2}} \sim \int_n^{\infty} x^{-1} (\log x)^{-\frac{9}{2}} dx, \quad n \to \infty .$$

If

$$\varphi_n = \frac{(n+1)^3 (\log(n+1))^3 (\log n)^5}{n^2} (\chi_n - \chi_{n+1}),$$

where

$$\chi_n = \frac{n^3}{(\log n)^7} \left(1 - \left(\frac{n \log n}{(n+1)\log(n+1)} \right)^2 \right)^3 ,$$

then $\lim_{n\to\infty} \varphi_n = 56$ and $x_n = (n \log n)^{-2}$ is an exact solution of the equation (1.22). (iv) If $\eta > 9$, then $\sigma = \eta - 7 > \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$ so $J = \infty$. Therefore, by Corollary 1.1 the equation (1.22) does not have decreasing regularly varying solutions.

Example 1.2 Consider the difference equation

$$\Delta \left(-n^{\eta} \sqrt{\log n}\, (\Delta x_n)^2 \right) = \frac{n^{\eta-3} \varphi_n}{(\log n)^{19/6}} \sqrt[3]{x_{n+1}}, \quad n \ge 1, \tag{1.25}$$

where φ_n is a positive real-value sequence such that $\lim_{n\to\infty} \varphi_n = \delta$ and $\eta \ne 2$. Here, $p_n = n^{\eta}\sqrt{\log n}$, and $q_n = n^{\eta-3} \varphi_n (\log n)^{-19/6}$, so $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 3 = \eta - \alpha - 1$. Let $\eta < 2 = \alpha$. Using that

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j \right)^{\frac{1}{\alpha}} \sim \sum_{k=n}^{\infty} \sqrt{\frac{\varphi_n}{2-\eta} \frac{1}{k (\log k)^{11/6}}} < \infty, \quad n \to \infty,$$

and show that $\mathcal{F}$ has a fixed point by using Lemma 1.4. Namely, the operator $\mathcal{F}$ has the following properties:

(i) *Operator $\mathcal{F}$ maps $\mathcal{X}$ into itself*: Let $x \in \mathcal{X}$. Using (3.13), (3.14), (3.15) and (3.16), we get

$$(\mathcal{F}x)_n \leq K^{\frac{\beta}{\alpha}} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \leq 2K^{\frac{\beta}{\alpha}} X_n \leq K X_n, \quad n \geq n_0 .$$

and

$$(\mathcal{F}x)_n \geq \kappa^{\frac{\beta}{\alpha}} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \geq \kappa^{\frac{\beta}{\alpha}} \frac{X_n}{2} \geq \kappa X_n, \quad n \geq n_0 .$$

This shows that $(\mathcal{F}x)_n \in \mathcal{X}$, for all $n \geq n_0$, that is, $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

(ii) *Operator $\mathcal{F}$ is increasing*, i.e. for any $x, y \in \mathcal{X}$, $x \leq y$ implies $\mathcal{F}x \leq \mathcal{F}y$.

Thus all the hypotheses of Lemma 1.4 are fulfilled implying the existence of a fixed point $x \in \mathcal{X}$ of $\mathcal{F}$, satisfying

$$x_n = \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{\frac{1}{\alpha}}, \qquad n \geq n_0 . \tag{3.17}$$

It is clear in view of (3.15) and the fact that $X_n \to 0$, $n \to \infty$, that x is a positive solution of (*E*) which satisfies $x_n \to 0$, $n \to \infty$. Moreover, due to (3.10), (3.13) and (3.15), we have

$$p_n(-\Delta x_n)^{\alpha} \leq K^{\beta} \sum_{k=n}^{\infty} q_k X_{k+1}^{\beta} \leq m \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} f_k, \tag{3.18}$$

where

$$f_k = \left(\frac{\omega_k{}^{\alpha}}{\xi_k{}^{\beta}} \right)^{\frac{1}{\alpha-\beta}}, \quad f = \{f_k\} \in \mathcal{SV} \quad \text{and} \quad m = K^{\beta} \lambda^{-\frac{\beta}{\alpha-\beta}}.$$

Since, $\eta < \alpha$ and (3.4) as well as $\eta > \alpha$ and (3.6) imply that $\sigma + \rho\beta < -1$, from (3.18) we conclude that $x_n^{[1]} \to 0$, $n \to \infty$, that is $x \in \mathbb{M}_{0,0}^{-}$. □

Theorem 3.4. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. If (3.5) holds, then there exists $x \in \mathbb{M}_{0,0}^{-}$.*

Proof. Suppose (3.5) holds. Define sequences $T = \{T_n\}$ and $G = \{G_n\}$ by

$$G_n = \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}}, \quad T_n = \left(\frac{\alpha-\beta}{\alpha} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j \right)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}}, \quad n \geq 1 . \tag{3.19}$$

Since the first condition from (3.5) implies $\sigma < -1$, application of Theorem 2.9 gives

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j \right)^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}}, \quad n \to \infty,$$

so that

$$T_n \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} G_n^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty$$

by Theorem 1.6 the equation (1.25) has a nontrivial slowly varying solution and any such solution x has the asymptotic representation

$$x_n \sim \left(\frac{\delta}{2-\eta} \right)^{\frac{3}{5}} \cdot (\log n)^{-1}, \quad n \to \infty.$$

If

$$\varphi_n = n^3 \left(\frac{\log n}{\log(n+1)} \right)^{\frac{19}{6}} \left[\frac{(\log \frac{n+1}{n})^2}{(\log n)^{\frac{3}{2}} (\log(n+1))^{\frac{1}{2}}} - \left(\frac{n+1}{n} \right)^{\eta} \left(\frac{\log \frac{n+2}{n+1}}{\log(n+2)} \right)^2 \right],$$

then $\delta = 2 - \eta$ and considered equation has an exact solution $x_n = (\log n)^{-1}$, $x \in ntr - \mathcal{SV}$. Notice that in the case $\eta > 2 = \alpha$, since $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, using Corollary 1.3, we conclude that $\mathcal{R}^{-} = \emptyset$.

OK, that would be all for today. Please feel free to ask questions and make comments.

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Clearly, $G \in \mathcal{SV}$ and $T \in \mathcal{SV}$. Applying Theorem 2.9-(ii) and using the first condition from (3.5) we get

$$\sum_{k=n}^{\infty} q_k T_{k+1}^{\beta} \sim \frac{1}{(\alpha-\eta)^{\frac{\alpha}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} n^{\eta-\alpha} \omega_n G_n^{\frac{\alpha\beta}{\alpha-\beta}},\ n \to \infty.$$

Thus, by Theorem 2.8, the previous relation gives

$$\begin{aligned} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j T_{j+1}^{\beta}\right)^{\frac{1}{\alpha}} &\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}} G_k^{\frac{\beta}{\alpha-\beta}} \\ &\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} (-\Delta G_k) \cdot G_k^{\frac{\beta}{\alpha-\beta}} \\ &\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} G_n^{\frac{\alpha}{\alpha-\beta}} \sim T_n, \quad n \to \infty. \end{aligned}$$

Consequently, we conclude that T satisfies the asymptotic relation (3.11).

The rest of the proof is the same as the proof of Theorem 3.3 where X_n is replaced with T_n. Then, a solution x of the equation (E) satisfying $\kappa T_n \le x_n \le K T_n$, for large n, is obtained by the application of Knaster-Tarski fixed point theorem and belongs to the class $\mathbb{M}_{0,0}^{-}$. $\square$

Theorem 3.5. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. If (3.7) holds, then there exists $x \in \mathbb{M}_{0,0}^{-}$.*

Proof. Suppose (3.7) holds. Using (3.1) and the assumption (3.7), we have that

$$\sum_{k=1}^{\infty} q_k \left(\sum_{j=k}^{\infty} \frac{1}{p_k^{1/\alpha}}\right)^{\beta} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} \sum_{k=1}^{\infty} k^{\beta} q_k p_k^{-\frac{\beta}{\alpha}} = \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} \sum_{k=1}^{\infty} k^{-1} \omega_k \xi_k^{-\frac{\beta}{\alpha}},\ \ n \to \infty\,.$$

Define sequences $Y = \{Y_n\}$ and $W = \{W_n\}$ by

$$W_n = \sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^{-\frac{\beta}{\alpha}}, \quad Y_n = \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}} W_n^{\frac{1}{\alpha-\beta}},\ \ n \ge 1\,. \tag{3.20}$$

Note that $W \in \mathcal{SV}$ and since $n\, p_n^{-\frac{1}{\alpha}} = n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}}$, we see that $Y \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Thus, application of Theorem 2.8 gives

$$\begin{aligned} \sum_{k=n}^{\infty} q_k Y_{k+1}^{\beta} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^{-\frac{\beta}{\alpha}} W_k^{\frac{\beta}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} (-\Delta W_k) \cdot W_k^{\frac{\beta}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} W_n^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty, \end{aligned}$$

which yields with the help of Theorem 2.9-(ii)

$$\begin{aligned} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j Y_{j+1}^{\beta}\right)^{\frac{1}{\alpha}} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \xi_k^{-\frac{1}{\alpha}} W_k^{\frac{1}{\alpha-\beta}} \\ &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} W_n^{\frac{1}{\alpha-\beta}} = Y_n, \quad n \to \infty. \end{aligned}$$

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Therefore, $Y = \{Y_n\}$ satisfies the asymptotic relation (3.11). Then, proceeding exactly as in the proof of Theorem 3.3, replacing X_n with Y_n, a solution x satisfying $\kappa Y_n \le x_n \le K Y_n$, for large n, is obtained by the application of Knaster-Tarski fixed point theorem, and belongs to a class $\mathbb{M}^-_{0,0}$. □

Proof of Theorem 3.2:
(i) Follows from Theorem 3.3-(i) and Theorem 3.4.
(ii) Follows from Theorem 3.3-(ii) and Theorem 3.5. □

3.2. *Asymptotic representation of strongly decreasing $\mathcal{RV}$-solutions*

To simplify the "only if" part of the proof of main results we prove the next two lemmas.

Lemma 3.6. *Let $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le 0$ only one of the following two statements holds:*
(i) $\rho = 0$ *and*

$$x_n \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{3.21}$$

Then, it is $\sigma = \eta - \alpha - 1 < -1$.
(ii) ρ *is given by* (3.9) *and*

$$x_n \sim \left[\frac{n^{\alpha+1} p_n^{-1} q_n}{(-\rho)^{\alpha}(\alpha-\eta-\rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.22}$$

Then, it is $\sigma < \eta - \alpha - 1$.

Proof. Suppose that (E) has a solution $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le 0$, satisfying $x_n > 0$, $\Delta x_n < 0$ for $n \ge n_0 + 1 \ge 2$ and expressed with (3.2). Summing (E) for $k \ge n \ge n_0$, we get

$$p_n(-\Delta x_n)^{\alpha} = \sum_{k=n}^{\infty} q_k x_{k+1}^{\beta},$$

which yields, using (3.1) and (3.2)

$$p_n(-\Delta x_n)^{\alpha} \sim \sum_{k=n}^{\infty} q_k x_k^{\beta} = \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k l_k^{\beta}, \quad n \to \infty. \tag{3.23}$$

The fact that $x_n^{[1]} = p_n(-\Delta x_n)^{\alpha} \to 0$ as $n \to \infty$ implies

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k l_k^{\beta} = 0,$$

so it must be $\sigma + \rho\beta \le -1$. We first consider the case $\sigma + \rho\beta = -1$. Then,

$$p_n(-\Delta x_n)^{\alpha} \sim \sum_{k=n}^{\infty} k^{-1} \omega_k l_k^{\beta} = \Omega_n, \quad n \to \infty, \tag{3.24}$$

where $\Omega = \{\Omega_n\} \in \mathcal{SV}$ and $\Omega_n \to 0$, $n \to \infty$. Consequently

$$-\Delta x_n \sim \left(\frac{\Omega_n}{p_n}\right)^{\frac{1}{\alpha}} = n^{-\frac{\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} \Omega_n^{\frac{1}{\alpha}}, \quad n \to \infty.$$

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Since $\lim_{n\to\infty} x_n = 0$, summing previous relation from n to ∞, we get

$$x_n \sim \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \left(\frac{\Omega_k}{\xi_k}\right)^{\frac{1}{\alpha}}, \quad n\to\infty\,, \tag{3.25}$$

implying that $1-\frac{\eta}{\alpha}\le 0$ i.e. $\eta\ge\alpha$ which is contradiction, so this case is impossible.

Therefore, $\sigma+\rho\beta<-1$. An application of Theorem 2.9-(ii) in (3.23) gives

$$-\Delta x_n = \left(\frac{1}{p_n}\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}\right)^{\frac{1}{\alpha}} \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}}\omega_n^{\frac{1}{\alpha}}\xi_n^{-\frac{1}{\alpha}} l_n^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}}, \quad n\to\infty. \tag{3.26}$$

Because $x_n\to 0$, $n\to\infty$, summing (3.26) from n to ∞ we get

$$x_n \sim \sum_{k=n}^{\infty}\frac{k^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}}\omega_k^{\frac{1}{\alpha}}\xi_k^{-\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}}, \quad n\to\infty. \tag{3.27}$$

From the last relation we conclude that it must be $(\sigma+\rho\beta+1-\eta)/\alpha\le -1$, so we distinguish two possibilities:

$$\text{(a)}\ \ \frac{\sigma+\rho\beta+1-\eta}{\alpha}=-1, \qquad \text{(b)}\ \ \frac{\sigma+\rho\beta+1-\eta}{\alpha}<-1\,. \tag{3.28}$$

If (a) holds, then $\sigma+\rho\beta+1=\eta-\alpha$. From (3.27), we get that (3.21) holds, and according to Theorem 2.9-(iii), $x\in\mathcal{SV}$. Thus, $\rho=0$ and (a) implies that $\sigma=\eta-\alpha-1$. On the other hand, if (b) holds, from (3.27), by Theorem 2.9-(ii), we obtain

$$x_n \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1}\omega_n^{\frac{1}{\alpha}}\xi_n^{-\frac{1}{\alpha}} l_n^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}\Big(-\frac{\sigma+\rho\beta+1-\eta}{\alpha}-1\Big)}, \quad n\to\infty. \tag{3.29}$$

Thus it must be

$$\rho=\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1, \tag{3.30}$$

implying that the regularity index of x is given by (3.9). Combined this with the assumption $\rho<0$, we get that $\sigma<\eta-\alpha-1$. Moreover, using (3.9) i.e. (3.30), we obtain

$$(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}\Big(-\frac{\sigma+\rho\beta+1-\eta}{\alpha}-1\Big)=\Big((\alpha-\eta-\rho\alpha)(-\rho)^{\alpha}\Big)^{\frac{1}{\alpha}}, \tag{3.31}$$

and

$$n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1}\xi_n^{-\frac{1}{\alpha}}\omega_n^{\frac{1}{\alpha}} l_n^{\frac{\beta}{\alpha}}=\Big(n^{\alpha+1}p_n^{-1}q_n\Big)^{\frac{1}{\alpha}} x_n^{\frac{\beta}{\alpha}}\,. \tag{3.32}$$

Then, from (3.29) we obtain that the asymptotic representation of x is given by (3.22). □

Lemma 3.7. *Let $p\in\mathcal{RV}(\eta)$, $\eta>\alpha$ and $q\in\mathcal{RV}(\sigma)$. For any $x\in\mathbb{M}_{0,0}^{-}\cap\mathcal{RV}(\rho)$ with $\rho\le\frac{\alpha-\eta}{\alpha}$ only one of the following two statements holds:*

(i) $\rho=\dfrac{\alpha-\eta}{\alpha}$ *and*

$$x_n\sim\frac{\alpha}{\eta-\alpha}n^{\frac{\alpha-\eta}{\alpha}}\xi_n^{-\frac{1}{\alpha}}\left(\sum_{k=n}^{\infty}k^{-1}\omega_k l_k^{\beta}\right)^{\frac{1}{\alpha}}, \ n\to\infty; \tag{3.33}$$

Then, it is $\sigma=\beta\,\dfrac{\eta-\alpha}{\alpha}-1$.

(ii) ρ *is given by* (3.9) *and* (3.22) *holds. Then, it is* $\sigma<\beta\,\dfrac{\eta-\alpha}{\alpha}-1$.

Proof. Suppose that (E) has a solution $x\in\mathbb{M}_{0,0}^{-}\cap\mathcal{RV}(\rho)$ with $\rho\le\frac{\alpha-\eta}{\alpha}$, satisfying $x_n>0$, $\Delta x_n<0$ for $n\ge n_0+1\ge 2$ and expressed with (3.2). Using (3.1) and (3.2) we have (3.23). As in the proof of previous lemma, the fact that $x_n^{[1]}=p_n(\Delta x_n)^{\alpha}\to 0$ as $n\to\infty$ implies that $\sigma+\rho\beta\le -1$.

If $\sigma+\rho\beta=-1$, then as in the proof of previous lemma we get (3.25), where Ω_n is given in (3.24). Using that $\eta>\alpha$, application of Theorem (2.9)-(ii) in (3.25) gives (3.33). Thus, $\rho=\frac{\alpha-\eta}{\alpha}$, implying that $\sigma=\beta\,\frac{\eta-\alpha}{\alpha}-1$.

Next we consider the case $\sigma+\rho\beta<-1$. An application of Theorem 2.9-(ii) in (3.23) give us (3.27) implying as previously two possibilities (a) or (b) in (3.28). However, the case (a) is not possible, because $\sigma+\rho\beta<-1$ implies

$$-1=\frac{\sigma+\rho\beta+1-\eta}{\alpha}<-\frac{\eta}{\alpha}\,,$$

which is a contradiction with $\eta>\alpha$. Thus, only (b) in (3.28) can be valid and so from (3.27), as previously, we obtain that ρ is given by (3.9) and x satisfies (3.22). Since, $\rho<\frac{\alpha-\eta}{\alpha}$ from (3.9) we conclude that $\sigma<\frac{\beta\eta}{\alpha}-\beta-1$. □

Now, we are in a position to prove the main results.

Theorem 3.8. *Suppose that $p\in\mathcal{RV}(\eta)$ and $q\in\mathcal{RV}(\sigma)$.*

(i) *Let $\eta<\alpha$. Equation (E) possesses regularly varying solutions x of index $\rho<0$ if and only if* (3.4) *holds.*

(ii) *Let $\eta>\alpha$. Equation (E) possesses regularly varying solutions x of index $\rho<\frac{\alpha-\eta}{\alpha}$ if and only if* (3.6).

In both cases ρ is given by (3.9) *and the asymptotic behavior of any such solution x is governed by the unique formula* (3.22).

Proof. **The "only if" part:** Suppose that $\eta<\alpha$ and $x\in\mathcal{RV}(\rho)$ with $\rho<0$. According to Theorem 2.7-(v) and (vi), $x\in\mathbb{M}^{-}$ and $\lim_{n\to\infty}x_n=0$. It is easy to prove (see [6, Lemma 3]) that if $S=\infty$, then for any solution in the class $\mathbb{M}^{-}$, it holds $\lim_{n\to\infty}x_n{}^{[1]}=0$. Thus, $x\in\mathbb{M}_{0,0}^{-}$. Then, it is clear that only the case (ii) of Lemma 3.6 is admissible for x. Thus, the regularity index of x is given by (3.9) and σ satisfies (3.4).

Suppose that $\eta>\alpha$ and $x\in\mathcal{RV}(\rho)$ with $\rho<\frac{\alpha-\eta}{\alpha}$. Since $\rho<0$ as previously we conclude that $x\in\mathbb{M}_{0}^{-}$. Therewith, in view of (3.3), by Theorem 2.7-(vi) we get

$$\lim_{n\to\infty}\frac{x_n}{\pi_n}=\frac{\eta-\alpha}{\alpha}\lim_{n\to\infty}n^{\varrho-\frac{\alpha-\eta}{\alpha}}l_n\xi_n^{\frac{1}{\alpha}}=0,$$

implying that $x\in\mathbb{M}_{0,0}^{-}$. It is clear that only the case (ii) of Lemma 3.7 is admissible for x, implying that the regularity index of x is given by (3.9) and that (3.6) holds.

From Lemmas 3.6 and 3.7 we obtain that the asymptotic representation of regularly varying solution x of index ρ is given by (3.22) in each of two cases (i) and (ii).

The "if" part: We perform the simultaneous proof for both of the cases. From Theorem 3.3 follows the existence of a solution $x\in\mathbb{M}_{0,0}^{-}$. It remains to prove that x satisfying (3.15) and (3.17) is a regularly varying sequence of index ρ. From (3.15) we have

$$0<\liminf_{n\to\infty}\frac{x_n}{X_n}\le\limsup_{n\to\infty}\frac{x_n}{X_n}<\infty,$$

where X_n is given by (3.8). Application of Lemma 1.2, using (3.11) and (3.17), yields

$$\begin{aligned} L \;&=\; \limsup_{n\to\infty}\frac{x_n}{X_n}\le\limsup_{n\to\infty}\frac{\Delta x_n}{\Delta X_n}=\limsup_{n\to\infty}\frac{-\Big(\frac{1}{p_k}\sum_{k=n}^{\infty}q_k x_{k+1}^{\beta}\Big)^{1/\alpha}}{-\Big(\frac{1}{p_k}\sum_{k=n}^{\infty}q_k X_{k+1}^{\beta}\Big)^{1/\alpha}}\\ &\le\; \left(\limsup_{n\to\infty}\frac{\sum_{k=n}^{\infty}q_k x_{k+1}^{\beta}}{\sum_{k=n}^{\infty}q_k X_{k+1}^{\beta}}\right)^{1/\alpha}\le\left(\limsup_{n\to\infty}\frac{-q_n x_{n+1}^{\beta}}{-q_n X_{n+1}^{\beta}}\right)^{1/\alpha}\\ &\le\; \left(\limsup_{n\to\infty}\frac{x_{n+1}}{X_{n+1}}\right)^{\beta/\alpha}=L^{\frac{\beta}{\alpha}}. \end{aligned}$$

Since $\beta < \alpha$, from above we conclude that

$$0 < L \leq 1\,. \tag{3.34}$$

Similarly, we can see that $l = \liminf_{n\to\infty} x_n/X_n$ satisfies

$$1 \leq l < \infty\,. \tag{3.35}$$

From (3.34) and (3.35) we obtain that $l = L = 1$, which means that $x_n \sim X_n$, $n \to \infty$ and ensures that x is a regularly varying solution of (E) with requested regularity index and the asymptotic representation given by (3.22). □

Theorem 3.9. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^-_{0,0} \cap ntr - \mathcal{SV}$ if and only if* (3.5) *holds. All such solutions of (E) enjoy the precise asymptotic formula*

$$x_n \sim \left[\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty. \tag{3.36}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^-_{0,0} \cap ntr - \mathcal{SV}$. Then, clearly only the statement (i) of Lemma 3.6 could hold. Therefore, $\rho = 0$, $\sigma = \eta - \alpha - 1$ and x satisfies (3.21). Then, since $\sigma < -1$, application of Theorem 2.9 gives

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}}\sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}}, \quad n \to \infty, \tag{3.37}$$

where we used that $\sigma + 1 = \alpha - \eta$. Denote

$$z_n = \sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}. \tag{3.38}$$

From Theorem 2.9-(iii) clearly $z = \{z_n\} \in \mathcal{SV}$ and (3.21) becomes

$$x_n = l_n \sim \frac{z_n}{(\alpha-\eta)^{\frac{1}{\alpha}}}, \quad n \to \infty. \tag{3.39}$$

From (3.38) and (3.39) we obtain the asymptotic relation

$$z_n^{-\frac{\beta}{\alpha}}(-\Delta z_n) \sim \frac{n^{-1}\xi_n^{-\frac{1}{\alpha}}\omega_n^{\frac{1}{\alpha}}}{(\alpha-\eta)^{\frac{\beta}{\alpha^2}}}, \quad n \to \infty. \tag{3.40}$$

By (3.39), we have that $z_n \to 0$, $n \to \infty$ and clearly $\{z_n\}$ is strictly decreasing. Summing (3.40) from n to ∞, using Theorem 2.8 and (3.37), we obtain

$$\frac{\alpha}{\alpha-\beta} z_n^{1-\frac{\beta}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{\beta}{\alpha^2}}}\sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{\beta-\alpha}{\alpha^2}}}\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}, \; n \to \infty. \tag{3.41}$$

Because $1 - \frac{\beta}{\alpha} > 0$, $z_n^{1-\frac{\beta}{\alpha}} \to 0$, $n \to \infty$, so (3.41) yields that the second condition in (3.5) is satisfied as well as that the asymptotic expression for x is

$$x_n \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}}\left(\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}} \sim \left(\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}},$$

when $n \to \infty$. This completes the "only if" part of the proof of Theorem 3.9.

The "if" part: From Theorem 3.4 we have the existence of a solution $x \in \mathbb{M}^-_{0,0}$. In the same way as in the proof of Theorem 3.8, replacing X_n with T_n given by (3.19) and with the application of Lemma 1.2 we obtain that $x_n \sim T_n$, $n \to \infty$, implying that such a solution is slowly varying and enjoys the precise asymptotic behavior (3.36). □

Theorem 3.10. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$ if and only if* (3.7) *holds. All such solutions of (E) enjoy the precise asymptotic behaviour*

$$x_n \sim \left(\alpha^{\alpha-1}\frac{\alpha-\beta}{(\eta-\alpha)^{\alpha}}\right)^{\frac{1}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}}\left[\sum_{k=n}^{\infty} k^{\beta} q_k p_k^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.42}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Then, clearly only the statement (i) of Lemma 3.7 could hold. Therefore, $\rho = \frac{\alpha-\eta}{\alpha}$, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and x satisfies (3.33). From (3.2) and (3.33) we get

$$l_n \sim \frac{\alpha}{\eta-\alpha}\xi_n^{-\frac{1}{\alpha}}\Omega_n^{\frac{1}{\alpha}}, \; n \to \infty, \tag{3.43}$$

where Ω_n is given in (3.24). From (3.24), we conclude that $\Omega \in \mathcal{SV}$, $\Omega_n \to 0$ as $n \to \infty$ and $\{\Omega_n\}$ is strictly decreasing. We transform (3.43) into the asymptotic relation for Ω

$$\Omega_n^{-\frac{\beta}{\alpha}}\Delta\Omega_n \sim -\left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} n^{-1}\omega_n\xi_n^{-\frac{\beta}{\alpha}} = -\left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} n^{\beta} q_n p_n^{-\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{3.44}$$

Summing (3.44) from n to ∞ and using Theorem 2.8 we obtain

$$\frac{\alpha}{\alpha-\beta}\Omega_n^{1-\frac{\beta}{\alpha}} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta}\sum_{k=n}^{\infty} k^{\beta} q_k p_k^{-\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{3.45}$$

Because $\Omega_n^{1-\frac{\beta}{\alpha}} \to 0$ as $n \to \infty$, (3.45) yields that the second condition in (3.7) is satisfied. The asymptotic expression (3.33) for x becomes

$$x_n \sim \frac{\alpha}{\eta-\alpha} n^{\frac{\alpha-\eta}{\alpha}}\xi_n^{-\frac{1}{\alpha}}\Omega_n^{\frac{1}{\alpha}} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}}\left[\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty} k^{\beta} q_k p_k^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

This completes the "only if" part of the proof of Theorem 3.10.

The "if" part: From Theorem 3.5 we obtain the existence of a solution $x \in \mathbb{M}^-_{0,0}$, while application of Lemma 1.2 as in the proof of Theorem 3.8, with Y_n instead of X_n, where Y_n is given by (3.20), proves that $x_n \sim Y_n$, $n \to \infty$, so that such a solution is in fact a $\mathcal{RV}$−solution of index $\frac{\alpha-\eta}{\alpha}$, with the precise asymptotic behavior given by (3.42). □

4. Corollaries and examples

In the previous section, we have shown that the existence of strongly decreasing $\mathcal{RV}$−solutions for the equation (E) with $\mathcal{RV}$ coefficients is fully characterized by the assumption $I < \infty$ if $S = \infty$ and by the assumption $J < \infty$ if $S < \infty$. In fact, the following corollary holds.

Corollary 4.1. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta \neq \alpha$ and $q \in \mathcal{RV}(\sigma)$.*

(i) Let $S = \infty$. Equation (E) has strongly decreasing $\mathcal{RV}$-solutions if and only if $I < \infty$.
(ii) Let $S < \infty$. Equation (E) has strongly decreasing $\mathcal{RV}$-solutions if and only if $J < \infty$.

Moreover, if $S = \infty$, then $J = \infty$ so by Theorem 1.1 $\mathbb{M}^-_{0,l} = \emptyset$. Otherwise, if $S < \infty$, denoting the series $Q = \sum_{k=1}^{\infty} q_k$, we have two cases:

(a) If $Q = \infty$, then $I = \infty$, so by Theorem 1.1 we have $\mathbb{M}^- = \mathbb{M}^-_0$ i.e. $\mathbb{M}^-_B = \emptyset$.
(b) If $Q < \infty$, then $I < \infty$, so by Theorem 1.1 we have $\mathbb{M}^- = \mathbb{M}^-_0 \cup \mathbb{M}^-_B$.

Using conclusions from the previous corollary and Theorem 1.1, we get the next two corollaries where we will use the following symbols:
* $\mathcal{R}$ denote the set of all regularly varying solutions,
* $\mathcal{R}^-$ denote the set of all decreasing regularly varying solutions,
* $\mathcal{R}^-_0 = \mathcal{R} \cap \mathbb{M}^-_0$.
* $\mathcal{R}^-_{0,0} = \mathcal{R} \cap \mathbb{M}^-_{0,0}$.

Corollary 4.2. *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S = \infty$. Then,*

$$\mathcal{R}^- = ntr - \mathcal{SV} \cup \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}^-_B$$

if and only if $I < \infty$.

Corollary 4.3. *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S < \infty$. Then,*

(i) If $\sigma < -1$ or $\sigma = -1$ and $Q < \infty$, then

$$\mathcal{R}^- = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_B.$$

(ii) If $\sigma = -1$ and $Q = \infty$ or $-1 < \sigma < \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R}^- = \mathcal{R}^-_0 = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}^-_{0,l}.$$

(iii) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J < \infty$, then

$$\mathcal{R}^- = \mathcal{R}^-_0 = \mathcal{RV}\left(\frac{\alpha - \eta}{\alpha}\right) \cup \mathbb{M}^-_{0,l}.$$

(iv) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J = \infty$ or $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R}^- = \emptyset.$$

Example 4.4. Consider the difference equation

$$\Delta\left(\frac{n^\eta}{\log n}\,(\Delta x_n)^3\right) = \frac{n^{\eta-7}\varphi_n}{\log^5 n}\,\sqrt{x_{n+1}{}^3}, \quad n \geq 1, \tag{4.1}$$

where φ_n is a positive real-value sequence such that $\lim_{n\to\infty} \varphi_n = \delta$ and $\eta \neq 3$. In this equation, $\alpha = 3$, $\beta = \dfrac{3}{2}$, $\{p_n\} \in \mathcal{RV}(\eta)$ and $\{q_n\} \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 7$.

(i) Suppose that $\eta < 3$. In this case

$$\sigma = \eta - 7 < \eta - 4 = \eta - \alpha - 1,$$

so in view of Theorem 3.8-(i) and Theorem 1.1 this equation has a strongly decreasing $\mathcal{RV}$-solution of index $\rho < 0$ as well as a solution in $\mathbb{M}^-_B = tr - \mathcal{SV}$. More precisely, by Theorem 3.8-(i) equation (4.1) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$. That solution has asymptotic behavior

$$x_n \sim \left(\frac{\delta}{8(9-\eta)}\right)^{\frac{2}{3}} n^{-2}\,(\log n)^{-\frac{8}{3}}, \quad n \to \infty. \tag{4.2}$$

If

$$\varphi_n = \frac{n^7 (n+1)^3}{(\log n)^4\,(\log(n+1))^5}\left((\log n)^9 \psi_n - (\log(n+1))^9 \left(\frac{n+1}{n}\right)^\eta \psi_{n+1}\right), \tag{4.3}$$

where

$$\psi_n = \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\left(\frac{\log n}{\log(n+1)}\right)^{\frac{8}{3}}\right)^3,$$

then $\delta = 8(9-\eta)$ and the considered equation has an exact solution $n^{-2}\,(\log n)^{-\frac{8}{3}}$.

(ii) For $\eta \in (3, 9)$ we have that $\eta > \alpha$ and $\sigma = \eta - 7 < \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$, so in view of Theorem 3.8-(ii) the equation (4.1) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$ and satisfies (4.2). This equation also possess a solution which belongs to a class $\mathbb{M}^-_{0,l}$.

(iii) Let $\eta = 9$. Then, $\sigma = 2 = \dfrac{\beta\eta}{\alpha} - \beta - 1$ and $J < \infty$. By Theorem 3.10 the equation (4.1) has a solution $x \in \mathcal{RV}(1 - \frac{\eta}{\alpha}) = \mathcal{RV}(-2)$ and any such solution x has the asymptotic representation

$$\begin{aligned} x_n &\sim \left(\frac{\delta}{16}\right)^{\frac{2}{3}} n^{-2}(\log n)^{\frac{1}{3}} \left(\sum_{k=n}^{\infty} k^{-1}(\log k)^{-\frac{9}{2}}\right)^{\frac{2}{3}} \\ &\sim \left(\frac{\delta}{16}\right)^{\frac{2}{3}} n^{-2}(\log n)^{\frac{1}{3}} \left(\frac{2}{7}\right)^{\frac{2}{3}} (\log n)^{-\frac{7}{3}} = \left(\frac{\delta}{56}\right)^{\frac{2}{3}} (n\,\log n)^{-2}, \quad n \to \infty, \end{aligned}$$

where we used that

$$\sum_{k=n}^{\infty} k^{-1}(\log k)^{-\frac{9}{2}} \sim \int_n^{\infty} x^{-1}(\log x)^{-\frac{9}{2}} dx, \quad n \to \infty\,.$$

If

$$\varphi_n = \frac{(n+1)^3(\log(n+1))^3(\log n)^5}{n^2}(\chi_n - \chi_{n+1}),$$

where

$$\chi_n = \frac{n^3}{(\log n)^7}\left(1 - \left(\frac{n \log n}{(n+1)\log(n+1)}\right)^2\right)^3,$$

then $\lim_{n\to\infty} \varphi_n = 56$ and $x_n = (n\,\log n)^{-2}$ is an exact solution of the equation (4.1).

(iv) If $\eta > 9$, then $\sigma = \eta - 7 > \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$ so $J = \infty$. Therefore, by Corollary 4.1 the equation (4.1) does not have decreasing regularly varying solutions.

Example 4.5. Consider the difference equation

$$\Delta\left(-n^\eta \sqrt{\log n}\,(\Delta x_n)^2\right) = \frac{n^{\eta-3}\,\varphi_n}{(\log n)^{19/6}}\,\sqrt[3]{x_{n+1}}, \quad n \geq 1, \tag{4.4}$$

where φ_n is a positive real-value sequence such that $\lim_{n\to\infty}\varphi_n = \delta$ and $\eta \neq 2$. Here, $p_n = n^\eta \sqrt{\log n}$, and $q_n = n^{\eta-3}\,\varphi_n\,(\log n)^{-19/6}$, so $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 3 = \eta - \alpha - 1$.

Let $\eta < 2 = \alpha$. Using that

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}} \sim \sum_{k=n}^{\infty}\sqrt{\frac{\varphi_n}{2-\eta}}\,\frac{1}{k\,(\log k)^{11/6}} < \infty, \quad n \to \infty,$$

by Theorem 3.10 the equation (4.4) has a nontrivial slowly varying solution and any such solution x has the asymptotic representation

$$x_n \sim \left(\frac{\delta}{2-\eta}\right)^{\frac{3}{5}} \cdot (\log n)^{-1}, \quad n \to \infty.$$

If

$$\varphi_n = n^3\left(\frac{\log n}{\log(n+1)}\right)^{\frac{19}{6}}\left[\frac{(\log\frac{n+1}{n})^2}{(\log n)^{\frac{3}{2}}(\log(n+1))^{\frac{1}{2}}} - \left(\frac{n+1}{n}\right)^{\eta}\left(\frac{\log\frac{n+2}{n+1}}{\log(n+2)}\right)^2\right],$$

then $\delta = 2 - \eta$ and considered equation has an exact solution $x_n = (\log n)^{-1},\ x \in ntr - \mathcal{SV}$.

Notice that in the case $\eta > 2 = \alpha$, since $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, using Corollary 4.3, we conclude that $\mathcal{R}^- = \emptyset$.

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