On the equivalence between Perov fixed point theorem and Banach contraction principle

Marija S. Cvetković

1 Introduction

Well-known Banach fixed point theorem, also known as Banach contraction principle, was a foundation for a development of metric fixed point theory and found applications in various areas. There were many generalizations of this result in the last years. We can observe two main directions in this area of research, including different contraction conditions or introducing analogous concept on different spaces such as partial, cone-metric, b-metric spaces, etc. Russian mathematician A. I. Perov [21] defined generalized cone metric space by defining a metric with values in $\mathbb{R}^n$. Then, this concept of metric space allowed him to define a new class of mappings, known as Perov contractions, which satisfy contractive condition similar to Banach's, but with a matrix $A \in \mathbb{R}^{n\times n}$ with nonnegative entries instead of constant q. This result found main application in the area of differential equations ([22, 28, 25]).

In [6] was presented extension of Perov theorem on a cone metric space, normal or solid. The concept of cone metric space (vector value metric space, K-metric space) has a long history (see [14, 26, 32]) and first fixed point theorems in cone metric spaces were obtained by Schröder [29, 30] in 1956. Cone metric space may be considered as a generalization of metric space and it is focus of the research in metric fixed point theory last few decades (see, e.g., [1, 2, 4], [9], [12], [15], [17], [27], [31] for more details). Concept of cone metric includes generalized metric in the sense of Perov, and contractive condition defined in [6] introduces a bounded linear operator instead of a matrix. Other requirements for this operator varies based on the ki

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1 Introduction

What do you think, how many new papers are published every year in SCIe journals? And what about Mathematics only? How can we be sure in correctness of those papers? I believe we have editors and reviewers for this purpose, but mistakes definitely happen. What is really hard to confirm is the novelty of presented results-even though it does not look the same it can easily turn out to be. The topic of today's talk is exactly that type of result but with a twist-the new interpretation of the well-known result has great impact in other areas of Mathematics and has significantly better performance. You have all heard for famous Banach contraction principle and its impact in different areas of Mathematics. On the other hand, Perov theorem and its extensions have been widely researched in the last decade in different settings. So now we have numerous results regarding common fixed point problem, coupled fixed point problem, multivalued mappings, Fisher contraction, Matkowski contraction and so on. What we intend to show today is that the extension of Perov theorem on normal cone metric space is equivalent to the famous Banach fixed point theorem.

Perov is a Russian mathematician who worked in the area of DEs and in 19[illegible]6. presented an interesting fixed point result as a tool for solving one type of DEs. It is interesting that he has presented only one more paper with the application of that fixed point theorem and there was no shown interest from scientific community for this result until 2000s. One of the first papers

This paper is focused on a relation between Banach and Perov theorem along with its generalizations on a complete normal cone metric space. Some equivalents between metric and normal cone metric spaces are presented and used to obtain different proof approach for several Perov type results.

2 Preliminaries

Some basic definitions and facts which are applied in subsequent sections are collected in this section. Since some correlations will be made, we give basic overview on generalized metric space in the sense of Perov, cone metric spaces and b-metric spaces.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 2.1. *([21]) A mapping $d : X \times X \mapsto \mathbb{R}^n$ is called a vector-valued metric on X if the following statements are satisfied for all $x, y, z \in X$.*

(d_1) $d(x,y) \geq 0_n$ and $d(x,y) = 0_n \Leftrightarrow x = y$, where $0_n = (0,\dots,0) \in \mathbb{R}^n$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$.

If $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$, $i = \overline{1,n}$.
Throughout this paper we denote by $\mathcal{M}_{n,n}$ the set of all $n \times n$ matrices, by $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative entries. We write Θ_n for the zero $n \times n$ matrix and I_n for the identity $n \times n$ matrix and further on we identify row and column vector in $\mathbb{R}^n$.
A matrix $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^m \to \Theta_n$, as $m \to \infty$.

Theorem 2.2. *(Perov [21, 22]) Let (X,d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ a matrix convergent to zero, such that*

$$d(f(x), f(y)) \leq A(d(x,y)), \quad x, y \in X.$$

Then:

(i) f has a unique fixed point $x^* \in X$;

on this topic, almost half of century later, was by Prof. Precup from Romania who gave emphasise on the imapct of Perov fixed point theorem and the faster convergence rate of the iterative sequence as its biggest advantage in comparison with Banach contraction principle.
The idea behind Perov theorem is in introducing matrix in a role of contractive constant. But first of all, we will change our perception of distance by introducing the concept of generalized metric space.

Let X be a nonempty set and $n \in \mathbb{N}$.

Definition 1.1. *([22]) A mapping $d : X \times X \mapsto \mathbb{R}^n$ is called a vector-valued metric on X if the following statements are satisfied for all $x, y, z \in X$.*

(d_1) $d(x,y) \geq 0_n$ and $d(x,y) = 0_n \Leftrightarrow x = y$, where $0_n = (0,\dots,0) \in \mathbb{R}^n$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$.

In order to understand the definition, note that if $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}^n$, then $x \leq y$ means that $x_i \leq y_i$, $i = \overline{1,n}$, so it is a type of lexicographic partial ordering.
Recall that $\mathcal{M}_{n,n}$ is the set of all $n \times n$ matrices, $\mathcal{M}_{n,n}(\mathbb{R}^+)$ the set of all $n \times n$ matrices with nonnegative entries. Also Θ_n is the zero $n \times n$ matrix and I_n is the identity $n \times n$ matrix. Do you know why can we identify row and column vector in $\mathbb{R}^n$?
If you recall the Banach contraction principle, the most important part is that the contractive constant q is between 0 and 1. Why so?
I have mentioned that we want to have now a matrix in a role of contractive constant, so we must see what does it mean that a matrix converges to zero. $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ is said to be convergent to zero if $A^m \to \Theta_n$, as $m \to \infty$.
At this moment, we have prepared the stage for a definition of Perov contraction aka generalized contraction and existence and uniqueness theorem for such mapping on a complete generalized metric space. (What does complete mean here?)

Theorem 1.2. *(Perov [22, 23]) Let (X,d) be a complete generalized metric space, $f : X \mapsto X$ and $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$ a matrix convergent to zero, such that*

$$d(f(x), f(y)) \leq A(d(x,y)), \quad x, y \in X.$$

Then:

(*ii*) *the sequence of successive approximations* $x_n = f(x_{n-1}), n \in \mathbb{N}$, *converges to* x^* *for any* $x_0 \in X$;

(*iii*) $d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}$;

(*iv*) *if* $g : X \mapsto X$ *satisfies the condition* $d(f(x), g(x)) \leq c$ *for all* $x \in X$ *and some* $c \in \mathbb{R}^n$, *then by considering the sequence* $y_n = g^n(x_0), n \in \mathbb{N}$, *one has*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

This result was extended on a setting of cone metric spaces.

Definition 2.3. *Let* E *be a real Banach space with a zero vector* θ. *A subset* P *of* E *is called a cone if:*

(*i*) P *is closed, nonempty and* $P \neq \{\theta\}$;

(*ii*) $a, b \in \mathbb{R}$, $a, b \geq 0$, *and* $x, y \in P$ *imply* $ax + by \in P$;

(*iii*) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ is the interior of P.

The cone P in a real Banach space E is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{2.1}$$

The least positive number satisfying (2.1) is called the normal constant of P. The cone P is called solid if $\operatorname{int} P \neq \emptyset$.

Definition 2.4. *[14] Let* X *be a nonempty set, and let* P *be a cone on a real ordered Banach space* E. *Suppose that the mapping* $d : X \times X \mapsto E$ *satisfies:*

(d_1) $\theta \leq d(x, y)$, *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(d_2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$, *for all* $x, y, z \in X$.

(*i*) f *has a unique fixed point* $x^* \in X$;

(*ii*) *the sequence of successive approximations* $x_n = f(x_{n-1}), n \in \mathbb{N}$, *converges to* x^* *for any* $x_0 \in X$;

(*iii*) $d(x_n, x^*) \leq A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}$;

(*iv*) *if* $g : X \mapsto X$ *satisfies the condition* $d(f(x), g(x)) \leq c$ *for all* $x \in X$ *and some* $c \in \mathbb{R}^n$, *then by considering the sequence* $y_n = g^n(x_0), n \in \mathbb{N}$, *one has*

$$d(y_n, x^*) \leq (I_n - A)^{-1}(c) + A^n(I_n - A)^{-1}(d(x_0, x_1)),\ n \in \mathbb{N}.$$

Compare this theorem with the Banach fixed point theorem? What are the similarities and what the differences? Do you think that those results are equivalent? I have mentioned Prof. Precup's paper, you can give it a look for more examples.
Since generalized metric space is just a special kind of cone metric space (normal or not?), it is natural to try to transfer this result in the setting of cone metric spaces. Have you heard about cone metric spaces? Cone is obviously well known term from the Geometry. If you think about it, it has a top which could be observed as zero vector, it is convex and when we apply symmetry to its axe we get disjoint cone only with same top. And therefore we have such definition of cone in Banach space.

Definition 1.3. *Let* E *be a real Banach space with a zero vector* θ. *A subset* P *of* E *is called a cone if:*

(*i*) P *is closed, nonempty and* $P \neq \{\theta\}$;

(*ii*) $a, b \in \mathbb{R}$, $a, b \geq 0$, *and* $x, y \in P$ *imply* $ax + by \in P$;

(*iii*) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ denotes $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ is the interior of P.
What happens when we look at the usual parial ordering on the real line?

Then d is called a cone metric on X and (X, d) is a cone metric space.

It is known that the class of cone metric spaces is bigger than the class of metric spaces. A lot of fixed point results, such as Banach contraction principle, are proved in the frame of cone metric spaces ([1, 2, 4], [12],[17, 18, 19]).

Suppose that E is a Banach space, P is a solid cone in E, whenever it is not normal, and $\leq$ is the partial order on E with respect to P.

Definition 2.5. *The sequence $\{x_n\} \subseteq X$ is convergent in X if there exists some $x \in X$ such that*

$$(\forall c \gg \theta)(\exists n_0 \in \mathbb{N})\, n \geq n_0 \implies d(x_n, x) \ll c.$$

We say that a sequence $\{x_n\} \subseteq X$ converges to $x \in X$ and denote that with $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Point x is called a limit of the sequence $\{x_n\}$.

Definition 2.6. *The sequence $\{x_n\} \subseteq X$ is a Cauchy sequence if*

$$(\forall c \gg \theta)(\exists n_0 \in \mathbb{N})\, n, m \geq n_0 \implies d(x_n, x_m) \ll c.$$

Every convergent sequence is a Cauchy sequence, but reverse do not hold. If any Cauchy sequence in a cone metric space (X, d) is convergent, then X is a complete cone metric space.

As proved in [14], if P is a normal cone, even in the case $\operatorname{int} P = \emptyset$, then $\{x_n\} \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$, $n \to \infty$. Similarly, $\{x_n\} \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, $n, m \to \infty$. Also, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $d(x_n, y_n) \to d(x, y)$, $n \to \infty$. Let us emphasise that this equivalences do not hold if P is a non-normal cone.

Perov generalized metric space is obviously a kind of a normal cone metric space. Defined partial ordering determines a normal cone $P = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = \overline{1, n}\}$ on $\mathbb{R}^n$, with the normal constant $K = 1$. Evidently, $A(P) \subseteq P$ if and only if $A \in \mathcal{M}_{n,n}(\mathbb{R}^+)$.

One of the results in [6] is a new generalization of Banach contraction principle in the sense of Perov.

Theorem 2.7. *Let (X, d) be a complete cone metric space with a solid cone P, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \leq A\left(d(x, y)\right), \quad x, y \in X. \tag{2.2}$$

Then:

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The cone P in a real Banach space E is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K \|y\|. \tag{1.1}$$

The least positive number satisfying this inequality is called the normal constant of P. The cone P is called solid if $\operatorname{int} P \neq \emptyset$.
The cone will be the base for new approach to understanding the distance. Distance is captured as nonnegative number, but now we will show another approach-it the values will be contained in a cone.

Definition 1.4. *[15] Let X be a nonempty set, and let P be a cone on a real ordered Banach space E. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:*

(d_1) $\theta \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is a cone metric space.

Do you know some examples of cone metric space? What about generalized metric space in the sense of Perov?

When we talk about cone metric spaces, we can also define convergence and the term of Cauchy sequence.

Definition 1.5. *The sequence $(x_n) \subseteq X$ is convergent in X if there exists some $x \in X$ such that*

$$(\forall c \gg \theta)(\exists n_0 \in \mathbb{N})\, n \geq n_0 \implies d(x_n, x) \ll c.$$

We say that a sequence $(x_n) \subseteq X$ converges to $x \in X$ and denote that with $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $n \to \infty$. Point x is called a limit of the sequence (x_n).

Definition 1.6. *The sequence $(x_n) \subseteq X$ is a Cauchy sequence if*

$$(\forall c \gg \theta)(\exists n_0 \in \mathbb{N})\, n, m \geq n_0 \implies d(x_n, x_m) \ll c.$$

(i) *f has a unique fixed point $z \in X$;*

(ii) *For any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z and*

$$d(x_n, z) \leq A^n(I - A)^{-1}(d(x_0, x_1)), \ n \in \mathbb{N};$$

(iii) *Suppose that $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in P$. Then if $y_n = g^n(x_0), n \in \mathbb{N}$, we have*

$$d(y_n, z) \leq (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)), \ n \in \mathbb{N}.$$

Furthermore, there was presented a similar result for normal cone metric space, but instead of the requirement of positiveness and $r(A) < 1$, only requirement is $K\|A\| < 1$ where K is a normal constant. Also, this normal cone is not necessarily solid.

Theorem 2.8. *Let (X, d) be a complete cone metric space, $d : X \times X \mapsto E$, P a normal cone with a normal constant K, $A \in \mathcal{B}(E)$ and $K\|A\| < 1$. If the condition (2.2) holds for a mapping $f : X \mapsto X$, then f has a unique fixed point $z \in X$ and the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z for any $x_0 \in X$.*

In Sections 11.3-11.5 of the classical monograph of Collatz [5] is given a general fixed point theorem in cone metric spaces, and in Section 12.1 it is considered a special case of this theorem. We note that that the first two parts of previous theorem can be obtained as a special case of Theorem 12.1 of [5].

Observe that with $\mathcal{B}(E)$ is denoted the set of all bounded linear operators on E and with $r(A)$ a spectral radius of an operator $A \in \mathcal{B}(E)$,

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n}.$$

If $r(A) < 1$, then the series $\sum_{n=0}^{\infty} A^n$ is absolutely convergent, $I - A$ is invertible in $\mathcal{B}(E)$ and

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}.$$

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Every convergent sequence is a Cauchy sequence, but reverse do not hold. If any Cauchy sequence in a cone metric space (X, d) is convergent, then X is a complete cone metric space.

If P is a normal cone, even in the case $\operatorname{int} P = \emptyset$, then $(x_n) \subseteq X$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$, $n \to \infty$. Similarly, $(x_n) \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, $n, m \to \infty$. Also, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $d(x_n, y_n) \to d(x, y)$, $n \to \infty$. Let us emphasise that this equivalences do not hold if P is a non-normal cone.

We will formulate Perov type theorem on cone met[ric s]paces since we have recalled all basic definitions.

Theorem 1.7. *Let (X, d) be a complete cone metric space with a solid cone P, $d : X \times X \mapsto E$, $f : X \mapsto X$, $A \in \mathcal{B}(E)$, with $r(A) < 1$ and $A(P) \subseteq P$, such that*

$$d(f(x), f(y)) \leq A\,(d(x, y))\,, \quad x, y \in X. \tag{1.2}$$

Then:

(i) *f has a unique fixed point $z \in X$;*

(ii) *For any $x_0 \in X$ the sequence $x_n = f(x_{n-1})$, $n \in \mathbb{N}$, converges to z and*

$$d(x_n, z) \leq A^n(I - A)^{-1}(d(x_0, x_1)), \ n \in \mathbb{N};$$

(iii) *Suppose that $g : X \mapsto X$ satisfies the condition $d(f(x), g(x)) \leq c$ for all $x \in X$ and some $c \in P$. Then if $y_n = g^n(x_0), n \in \mathbb{N}$, we have*

$$d(y_n, z) \leq (I - A)^{-1}(c) + A^n(I - A)^{-1}(d(x_0, x_1)), \ n \in \mathbb{N}.$$

In the discussion regarding normal cone metric space, we will only ask for A to have norm less than $\frac{1}{K}$. Even then we come up with the same conclusion: existence, uniqueness and convergence of the iterative sequence.

Let us go back, what was our basic idea-connection between Perov theorem on complete normal cone metric space and Banach theorem on complete metric space. We recalled all necessary basic definitions and notations and now we are ready to formulate our main result.

Theorem 1.8. *Perov theorem is equivalent the Banach fixed point theorem.*

Also, if $\|A\| < 1$, then $I - A$ is invertible and

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

If X is a Banach space with a cone P and operator $A : E \mapsto E$, then:

(*i*) A is a positive operator if $A(P) \subseteq P$;

(*ii*) A is an increasing operator if $x \leq y \implies A(x) \leq A(y)$, for any $x, y \in X$.

If $A \in \mathcal{B}(E)$, then (*i*) and (*ii*) are equivalent ([6]).
Omitting the boundedness condition, we obtain the following result:

Theorem 2.9. *Let (X, d) be complete cone metric space with a solid cone P and $f : X \mapsto X$ a continuous mapping. If there exists an increasing operator $A : E \mapsto E$ such that $\lim_{n\to\infty} A^n(e) = \theta$, $e \in E$, and, for any $x, y \in X$,*

$$d(f(x), f(y)) \leq A(d(x, y)), \tag{2.3}$$

then a mapping f has a unique fixed point in X.

Conditions of Theorem 2.9 could be less strict asking for A to be increasing and tend to zero only on P.

Recent results in cone metric fixed point theory established some relation between b-metric spaces and normal cone metric spaces.

Definition 2.10. *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to [0, +\infty)$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*

(b_1) *$d(x, y) = 0$ if and only if $x = y$;*

(b_2) *$d(x, y) = d(y, x)$;*

(b_3) *$d(x, z) \leq s[d(x, y) + d(y, z)]$.*

In this case, the pair (X, d) is called a b-metric space (with constant s).

Definitions of Cauchy and convergent sequence in a b-metric space, as well as completeness, go analogously as in a metric space.

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One direction is obvious, Banach contraction principle is a Perov type contraction since metric space is normal cone metric space with the cone $(0, \infty)$ and the contraction inequality is fulfilled.
Otherwise, the idea of the proof is to first show that any cone metric space is b-metric space and to combine that result with very well known fact that any cone metric space can be remetrizable in order to get normal constant K equal to 1.
Only difference between metric paces and b-metric spaces is in the triangle inequality. For b-metric space we have some constant $s \geq 1$ such that

$$d(x, z) \leq s[d(x, y) + d(y, z)]$$

. It is highly intuitive how to define a b-metric on a normal cone space. Assume (X, d) is a normal cone metric space, then

$$D(x, y) = \|d(x, y)\|, \quad x, y \in X \tag{1.3}$$

is a b-metric on X with a same constant as normal cone.
In order to prove this, we need to check all assumptions that hold for b-metric. It is pretty obvious that D is nonnegative, symmetric and that the distance is equal to zero only if $x = y$. Also,

$$D(x, y) = \|d(x, y)\| \leq K\left(\|d(x, z)\| + \|d(z, y)\|\right) = K\left(D(x, z) + D(z, y)\right).$$

Thus, (X, D) is a b-metric space.
If the normal constant K is equal to 1, then (X, D) is a metric space.
But we did not finish the job here since we need to consider what happens with convergence in relation to the new b-metric. Fortunately, (X, d) is a complete cone metric space if and only if (X, D) is a complete b-metric space, limit and convergence are also maintained.
In this way, there is a different approach to the proof of existence and uniqueness theorem for normal cone metric space through b-metric. You can find this approach in one of our papers published in AMC. As mentioned, the normal constant can be observed as equal to 1 since

Theorem 1.9. *Let (X, d) be a cone metric space, $P \subseteq E$ a normal cone with a normal constant K where $(E, \|\cdot\|)$ is a Banach space. Then:*

(i) A function $\|\cdot\|_1 : E \mapsto \mathbb{R}$ defined with

$$\|x\|_1 = \inf\{\|u\| \mid x \leq u\} + \inf\{\|v\| \mid v \leq x\}, \; x \in E,$$

is a norm on E.

3 Main results

There were several papers ([3, 11, 15]) studying relations between cone metric spaces in general, and especially normal cone metric spaces, on one, and metric spaces on the other side. Many efforts are made in the attempt of reduction any cone metric space to a metric space. In the case that (X, d) is a normal cone metric space with a normal constant K, we may introduce a b-metric as presented in several recent papers.

Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Define a function $D : X \times X \mapsto \mathbb{R}$,

$$D(x, y) = \|d(x, y)\|, \quad x, y \in X \tag{3.4}$$

Theorem 3.1. *A function D defined in (3.4) is a b-metric on X with a constant K.*

Proof. Let $x, y, z \in X$ be arbitrary points. From the definition of norm and (d_1) it easily follows that (b_1) holds. D is also a symmetric function since it directly follows from the symmetry of the norm. From the fact that d is a metric on X, (d_3) and since (X, d) is a normal cone metric space, we have

$$D(x, y) = \|d(x, y)\| \le K\left(\|d(x, z)\| + \|d(z, y)\|\right) = K\left(D(x, z) + D(z, y)\right).$$

Thus, (X, D) is a b-metric space. □

If the normal constant K is equal to 1, then (X, D) is a metric space.

However, if (X, d) is a complete normal cone metric space, $\{x_n\}$ is Cauchy sequence in (X, d) if and only if $\lim\limits_{n,m\to\infty} \|d(x_n, x_m)\| = 0$ and $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \|d(x_n, x)\| = 0$. Therefore, we may state the following corollary.

Theorem 3.2. *(X, d) is a complete cone metric space, P a normal cone with a normal constant K and D an b-metric defined as in (3.4) if and only if (X, D) is a complete b-metric space.*

We will give another proof of the generalization of Perov fixed point theorem in the setting of normal cone metric space.

(ii) Norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent norms on X.

(iii) If we observe P as a cone in Banach space $(E, \|\cdot\|_1)$, then (X, d) is a normal cone metric space with a normal constant equal to 1.

The equivalence of the norms allows us to determine the relation between $\|A\|$ and $\|A\|_1$.

Note that the setting has changed, first from the normal cone metric space to b-metric space based on the remetrization, and afterwards from b-metric space to metric space, thanks to the previous theorem. Completeness, Cauchyness and convergence are kept all through those metric and norm changes.

Hence, we got the desired result, Perov theorem on normal cone metric space is equivalent to well known Banach metric space.But, that does not mean that the fixed point theorem of Perov type on a normal cone metric space is worthless, on contrary. As Precup mentioned, the convergence rate is much faster if we apply Perov theorem instead of Banach theorem.

As an example, we can observe generalized metric space in the sense of Perov which is a normal cone metric space with a normal constant $K = 1$.

Example 1. Define a mapping $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with $f(x) = (\frac{x_1}{2} + x_2, \frac{x_2}{2})$, $x = (x_1, x_2) \in \mathbb{R}^2$. Let

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix},$$

then $\lim_{n\to\infty} A^n = \Theta_2$ and

$$d(f(x), f(y)) \le A(d(x, y)), \ x, y \in \mathbb{R}^2.$$

Since $\|A\| = 1$, $D(f(x), f(y)) \le D(x, y)$ and if $x = (0, 0)$, $y = (0, 1)$, it follows that f is not a contraction in $(\mathbb{R}^2, D)$, but it is a Perov contraction and based on Perov theorem it possesses the unique fixed point $(0, 0)$.

We can also formulate very important corollary, if f is a mapping on a complete metric space such that f^n is a Perov type contraction, than f possesses a unique fixed point.

On the other hand, Perov theorem on solid (non-normal)cone metric space with the assumption that $r(A) < 1$ could not be derived directly from Banach theorem. You can try to prove it on the following example:

Theorem 3.3. *Let (X,d) be a complete cone metric space, P a normal cone with a normal constant K and $f : X \mapsto X$ a self-mapping. If there exists an operator $A \in \mathcal{B}(E)$ such that $K\|A\| < 1$, for all $x, y \in X$,*

$$d(f(x), f(y)) \leq A(d(x,y)), \tag{3.5}$$

then f has a unique fixed point in X.

Proof. From the condition (3.5) and the fact that P is a normal cone, it follows

$$D(fx, fy) = \|d(f(x), f(y))\| \leq K\|A(d(x,y))\| \leq K\|A\|D(x,y), \ x, y \in X,$$

and f is a contraction in b-metric space and the existence of an unique fixed point follows by the generalization of Banach fixed point theorem in b-metric space presented in [7]. □

Observe that we can obtain the same result from Banach fixed point theorem (on complete metric spaces) by renorming, as presented in [13].

Theorem 3.4. *Let (X,d) be a cone metric space, $P \subseteq E$ a normal cone with a normal constant K where $(E, \|\cdot\|)$ is a Banach space. Then:*

(i) A function $\|\cdot\|_1 : E \mapsto \mathbb{R}$ defined with

$$\|x\|_1 = \inf\{\|u\| \mid x \leq u\} + \inf\{\|v\| \mid v \leq x\}, \ x \in E,$$

is a norm on E.

(ii) Norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent norms on X.

(iii) If we observe P as a cone in Banach space $(E, \|\cdot\|_1)$, then (X,d) is a normal cone metric space with a normal constant equal to 1.

The equivalence of the norms allows us to determine the relation between $\|A\|$ and $\|A\|_1$.

Remark 3.5. Based on the previously made observations regarding renorminization of a normal cone with a normal constant K and Theorem 3.3, we may conclude that existence of the unique fixed point Perov type contractions (including extended and more general contractive conditions) on normal cone metric spaces could be derived from analogous results on metric spaces.

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Example 2. Let c_0 be the set containing all sequences of real numbers convergent to zero equipped with supremum norm $\|\cdot\|_\infty$ and define $A : E \mapsto E$ with

$$A(x) = A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (0, x_3, \frac{x_4}{2}, \ldots, \frac{x_{n+1}}{2}, \ldots), \quad x = (x_n) \in c_0.$$

The requirement that A contains only positive entries, as stated in Perov theorem, could be removed thanks to the normality of the defined cone in generalized metric space. This could be explained also by the fact that, from the definition of matrix norm, only absolute value of matrix entries has impact on the norm value. So Perov type theorems are applicable, regardless of the positivity of matrix elements, if absolute value of all entries are less than 1.

The question that raises is why do we do this? What is the impact of these types of results? Perov theorem has a wide range of application and estimations obtained by Perov theorem and generalized metric are better than by using usual metric spaces and some well-known theorems. In comparison with Schauder, Krasnoselskii, Leray-Schauder and Banach theorem, Perov theorem has best results. We will discuss here only on Banach and Perov theorem and compare the convergence rate in those two cases.

Example 3. If (X,d) is a complete metric space and $T_i : X \times X \mapsto X$, $i = 1, 2$, solution of a system

$$\begin{aligned} T_1(x,y) &= x \\ T_2(x,y) &= y, \end{aligned} \tag{1.4}$$

is a fixed point of a mapping $T : X \times X \mapsto X \times X$ defined with

$$T(x,y) = (T_1(x,y), T_2(x,y)), \ x, y \in X.$$

To apply Banach theorem, T should be a contraction on $X \times X$. Let D be a metric on $X \times X$ induced by d, then

$$D(F(x,y), F(u,v)) \leq qD((x,y),(u,v)), \ (x,y),(u,v) \in X \times X,$$

for some $q \in (0,1)$.
If $D((x,y),(u,v)) = d(x,y) + d(u,v)$, $(x,y),(u,v) \in X \times X$, then

$$d(T_1(x,y), T_1(u,v)) + d(T_2(x,y), T_2(u,v)) \leq q(d(x,y) + d(u,v)), \tag{1.5}$$

Focusing on just first two statements of Perov theorem, we may state the following result:

Theorem 3.6. *Perov theorem is a consequence of a Banach fixed point theorem.*

Proof. Notice that generalized metric space introduced by Perov is a type of normal cone metric space.
If $P = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = \overline{1,n}\}$, then P evidently determines a cone in a Banach space $\mathbb{R}^n$ with supremum norm, $\|x\| = \max\limits_{i=1^n} |x_i|$, and $x \leq y$ if and only if $x_i \leq y_i$, $i = \overline{1,n}$. Since $\theta \leq x \leq y$, for $\theta = (0, 0, \dots, 0)$, $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, implies $0 \leq x_i \leq y_i$, $i = \overline{1,n}$, then $\|x\| = \max\limits_{i=\overline{1,n}} |x_i| \leq \max\limits_{i=\overline{1,n}} |y_i| = \|y\|$ and P is a normal cone with a normal constant $K = 1$.
By taking into the account results of Theorem 3.1, it follows that for any generalized metric space (X, d) in the sense of Perov, the appropriate b-metric space (X, D) is a metric space.
Assume that the requirements of Perov theorem are fulfilled for some $A \in \mathcal{M}_{m,m}(\mathbb{R}^+)$ such that $A^n \to \Theta_m$, as $n \to \infty$. Since a matrix A converges to the zero matrix, then $\|A^n\| \to 0$, $n \to \infty$. Choose $n_0 \in \mathbb{N}$ such that $\|A^n\| < 1$ for any $n \geq n_0$. For such n,

$$d(f^n x, f^n y) \leq A^n(d(x,y)), \quad x, y \in \mathbb{R}^m,$$

and

$$D(f^n x, f^n y) \leq \|A^n\| D(x,y), \quad x, y \in \mathbb{R}^m. \tag{3.6}$$

If we apply Banach contraction principle for f^n and $q = \|A^n\| < 1$, f^n has a unique fixed point z in X. Since $f^n(fz) = fz$, it must be $fz = z$. If $fu = u$ for some $u \in X$, then $f^n u = u$, so $u = z$.
Hence, Perov theorem is a direct consequence of Banach contraction principle.
It is easy to observe that the iterative sequence $\{x_n\}$ is a Cauchy sequence, thus convergent, and since $\{f^{nk(x)}\}_{k\in\mathbb{N}}$ converges to z by Banach fixed point theorem, (ii) holds. □

Remark 3.7. On the other hand, if $n = 1$, then generalized metric space is a metric space and a positive matrix $A = [q]$ tends to zero if and only if $q < 1$. Thus, Banach contraction principle is a Perov fixed point theorem

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for any $(x,y), (u,v) \in X \times X$, because of

$$d(T_i(x,y), T_i(u,v)) \leq \frac{q}{2}(d(x,y) + d(u,v)),\ \ i = 1, 2, \tag{1.6}$$

holds for any $(x,y), (u,v) \in X \times X$.
On the other hand, if Perov theorem would be applied, T_1 and T_2 should be such that

$$d(T_i(x,y), T_i(u,v)) \leq a_i d(x,u) + b_i d(y,v),\ \ (x,y),(u,v) \in X \times X, i = 1, 2,$$

for some nonnegative $a_i, b_i \geq 0$, $i = 1, 2$, and a matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

convergent to zero, so

$$a_1 + b_2 + \sqrt{-2a_1b_2 + 4a_2b_1 + {a_1}^2 + {b_2}^2} < 2.$$

If we try to use Banach fixed point theorem in various norms, then $\max\{a_1, a_2\}$, $\max\{b_1, b_2\}$ should be less than $\frac{1}{2}$, or $\max\{a_1, a_2\} + \max\{b_1, b_2\} < 1$. Anyway, this result is more strict than $r(A) < 1$ since

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{3} \end{bmatrix}$$

has spectral radius $r(A) = \frac{7}{9}$ less than 1, but previous estimations regarding the entries are not valid.

Perov fixed point theorem found application in solving various systems of differential equations. But, in some cases it is possible to replace it with the corollary regarding the f^n. I will just show you this example, we will not go through it, but if you need some clarification, we can discuss on it afterwards.

Example 4. Let (X_i, d_i), $i = \overline{1,m}$ be some complete metric spaces and define a generalized metric d on their Cartesian product $X = \prod\limits_{i=1}^{m} X_i$ and (Y, τ) be a for $n = 1$. However, remarks regarding distance presented in (iii) and (iv) (easily observed if we take $g = f$) could not be derived directly from Banach contraction principle since the inequality (3.6) do not imply (iii).

Example 1. Define a mapping $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with $f(x) = (\frac{x_1}{2} + x_2, \frac{x_2}{2})$, $x = (x_1, x_2) \in \mathbb{R}^2$. Let

$$A = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix},$$

then $\lim_{n\to\infty} A^n = \Theta_2$ and

$$d(f(x), f(y)) \leq A(d(x, y)), \ x, y \in \mathbb{R}^2.$$

Since $\|A\| = 1$, $D(f(x), f(y)) \leq D(x, y)$ and if $x = (0, 0)$, $y = (0, 1)$, it follows that f is not a contraction in $(\mathbb{R}^2, D)$, but it is a Perov contraction and based on Perov theorem it possesses a unique fixed point $(0, 0)$.

From the proof of Theorem 3.6 and the previous example, we may notice correlation between Perov theorem and well-known consequence of Banach theorem.

corollary 3.1. *Let (X, d) be a complete metric space, $f : X \mapsto X$ a mapping. If*

$$d(f^n(x), f^n(y)) \leq q d(x, y), \quad x, y \in X,$$

for some $n \in \mathbb{N}$ and $q \in [0, 1)$, then f has a unique fixed point in X.

The following example shows that Perov type theorems including requirement $r(A) < 1$ could not be derived directly from Banach theorem.

Example 2. Let c_0 be the set containing all sequences of real numbers convergent to zero equipped with supremum norm $\|\cdot\|_\infty$ and define $A : E \mapsto E$ with

$$A(x) = A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (0, x_3, \frac{x_4}{2}, \ldots, \frac{x_{n+1}}{2}, \ldots), \quad x = \{x_n\} \in c_0.$$

Operator A is linear on Banach space $(c_0, \|\cdot\|_\infty)$ and also bounded since $\|Ax\|_\infty \leq \|x\|_\infty$. By choosing $e_3 = (0, 0, 1, 0, \ldots, 0, \ldots) \in c_0$, it follows $\|A\| = 1$ by taking into account previous inequality.
For any $m \in \mathbb{N}$,

$$A^m(x) = A^m(x_1, x_2, x_3, \ldots) = (0, \frac{x_{m+2}}{2^{m-1}}, \frac{x_{m+3}}{2^m}, \ldots), \quad x = \{x_n\} \in c_0,$$

Hausdorff topological space.Of $f = (f_1, f_2) : X \times Y \mapsto X \times Y$ is an operator, consider the system of functional-differential equations:

$$\begin{aligned} x(t) &= \int_0^1 K(t, s, x(s), y(s))ds + g(t), \quad t \in [0, 1], \\ y(t) &= \int_0^1 H(t, s, x(s), y(s), y(y(s)))ds, \quad t \in [0, 1], \end{aligned}$$

where $x \in X$ and $y \in Y$, continuous mappings $K \in C([0, 1] \times [0, 1] \times \mathbb{R}^m \times [0, 1], \mathbb{R}^m)$, $g \in C([0, 1], \mathbb{R}^m)$ and $H \in C([0, 1] \times [0, 1] \times \mathbb{R}^m \times [0, 1] \times [0, 1], \mathbb{R})$,

Under assumptions that codomain of H is contained in $[0, 1]$, that H is a first coordinate Lipschitzian mapping with a constant L and K is a Perov generalized contraction, this system has at least one solution in $X \times Y$ for $X = C([0, 1], \mathbb{R}^m) = \prod_{i=1}^{m} X_i$, $X_i = C[0, 1]$, $i = \overline{1, m}$ and Y set of all Lipschitzian mappings on $C([0, 1], [0, 1])$ with a constant L. Observe that we could not use Banach theorem instead of Perov to obtain this conclusion due to the contractive condition for K.

In conclusion, we have shown that Perov theorem on normal cone metric space is equivalent to Banach (on solid not), but still its applications are of huge significance in the area of differential, operator and integral equations. What could be done in the future? Maybe the convergence rate should be compared with more details for the normal cone metric space, solid and non-normal cone metric space and metric space. Also it is possible to discuss on novelty of Perov type theorem on solid cone metric space in comparison with some other well-known theorems in the metric fixed point theory.
There is long list of references on this topic and if you are interested in the research in this area, I can point out to you the most important ones.
Thank you for your attention! Let me know if you have any questions.

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therefore, observing $e_{m+2} \in c_0$ with all zeros except one on (m+2)-nd place (i.e., $(e_{m+2})_n = \delta_{n,m+2}$, $n \in \mathbb{N}$), we obtain $\|A^m\| = \frac{1}{2^{m-1}}$. Spectral radius of A is $\frac{1}{2}$, A is a positive operator, so all the conditions of Theorem 2.7 are satisfied since

$$d(A(x), A(y)) \leq A(d(x, y)), \ x, t \in c_0,$$

where $\leq$ is usual partial ordering on c_0, i.e. $x_n \leq y_n$, $n \in \mathbb{N}$, determining a normal cone and $d : c_0 \times c_0 \mapsto c_0$ defined by $d(x, y)(n) = |x(n) - y(n)|$, $n \in \mathbb{N}$ is a cone metric.
On the other hand, since normal constant and $\|A\|$ are equal to 1, norm inequality implies

$$D(A(x), A(y)) \leq D(x, y),$$

thus Banach theorem is not applicable (let $x = \theta$ and $y = e_3$).

We may also assume that $K = 1$ due to the renormization and the invariance of spectral radius in renormized space. It is important to notice that $r(A) < 1$ implies $\|A^n\| < 1$ for some $n \in \mathbb{N}$, so instead of Banach theorem, we should consider Consequence 3.1.

If the inequality (2.2) holds, then, since A is an increasing operator,

$$d(f^n(x), f^n(y)) \leq A^n(d(x, y)),$$

thus,

$$D(f^n(x), f^n(y)) \leq \|A^n\|(d(x, y)),$$

and existence and uniqueness of a fixed point for a mapping f follows directly from Consequence 3.1.

In Example 1 f^3 is a contraction in induced metric space, and in Example 2 f^2.

As presented in [6], the requirement that A contains only positive entries, as stated in Perov theorem, could be removed thanks to the normality of the defined cone in generalized metric space. This could be explained also by the fact that, from the definition of matrix norm, only absolute value of matrix entries has impact on the norm value. So Perov type theorems are applicable, regardless of the positivity of matrix elements, if all entries are less than 1.

Perov theorem has a wide range of application and estimations obtained by Perov theorem and generalized metric are better than by using usual metric spaces and some well-known theorems. In [24] coupled fixed point

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problem on Banach space was analyzed and, implementation of various metric and vector-valued metric in the sense of Perov, lead to the conclusion that results obtained by Perov theorem are better and unify other results. The comparison is made for Schauder, Krasnoselskii, Leray-Schauder and Perov theorem. We will discuss results obtained by Banach fixed point theorem and compare them in the case of metric space.

Example 3. If (X, d) is a complete metric space and $T_i : X \times X \mapsto X$, $i = 1, 2$, solution of a system

$$\begin{aligned} T_1(x, y) &= x \\ T_2(x, y) &= y, \end{aligned} \tag{3.7}$$

is a fixed point of a mapping $T : X \times X \mapsto X \times X$ defined with

$$T(x, y) = (T_1(x, y), T_2(x, y)), \ x, y \in X.$$

To apply Banach theorem, T should be a contraction on $X \times X$. Let D be a metric on $X \times X$ induced by d, then

$$D(F(x, y), F(u, v)) \leq qD((x, y), (u, v)), \ (x, y), (u, v) \in X \times X,$$

for some $q \in (0, 1)$.
If $D((x, y), (u, v)) = d(x, y) + d(u, v)$, $(x, y), (u, v) \in X \times X$, then

$$d(T_1(x, y), T_1(u, v)) + d(T_2(x, y), T_2(u, v)) \leq q(d(x, y) + d(u, v)), \tag{3.8}$$

for any $(x, y), (u, v) \in X \times X$, because of

$$d(T_i(x, y), T_i(u, v)) \leq \frac{q}{2}(d(x, y) + d(u, v)), \ i = 1, 2, \tag{3.9}$$

holds for any $(x, y), (u, v) \in X \times X$.
On the other hand, if Perov theorem would be applied, T_1 and T_2 should be such that

$$d(T_i(x, y), T_i(u, v)) \leq a_i d(x, u) + b_i d(y, v), \ (x, y), (u, v) \in X \times X, i = 1, 2,$$

for some nonnegative $a_i, b_i \geq 0$, $i = 1, 2$, and a matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

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convergent to zero. This means that $r(A) < 1$ or, equivalently,

$$a_1 + b_2 + \sqrt{-2a_1b_2 + 4a_2b_1 + {a_1}^2 + {b_2}^2} < 2.$$

Considering (3.9), $\max\{a_1, a_2\}$, $\max\{b_1, b_2\}$ should be less than $\frac{1}{2}$, or in view of (3.8), $\max\{a_1, a_2\} + \max\{b_1, b_2\} < 1$. Anyway, this result is more strict than $r(A) < 1$.

If

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{3} \end{bmatrix},$$

then $r(A) = \frac{7}{9}$, but neither of the inequalities (3.8) and (3.9) is satisfied.

Perov fixed point theorem found application in solving various systems of differential equations. But, in some cases like [28], it is possible to replace it with the Consequence 3.1.

Example 4. Let (X_i, d_i), $i = \overline{1, m}$ be some complete metric spaces and define a generalized metric d on their Cartesian product $X = \prod_{i=1}^{m} X_i$ with

$$d(x, y) = \begin{bmatrix} d_1(x_1, y_1) \\ d_2(x_2, y_2) \\ \vdots \\ d_m(x_m, y_m) \end{bmatrix},$$

for $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in X$. As previously discussed, (X, d) is, as generalized metric space, also a normal cone metric space with a normal constant $K = 1$.
Let (Y, τ) be a Hausdorff topological space and $f = (f_1, f_2) : X \times Y \mapsto X \times Y$ an operator.Theorem 2.1 of [28] states that if f is continuous, (Y, τ) has a fixed point property (i.e., every continuous mapping $g : Y \mapsto Y$ has a fixed point) and there exists a matrix $S \in R^{m\times m}$ convergent to zero matrix such that

$$d(f_1(u, y), f_1(v, y)) \leq Sd(u, v), \quad u, v \in X, y \in Y, \tag{3.10}$$

then f has a fixed point. Uniqueness is not guaranteed because of contractive condition based on the first coordinate.

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Instead of using Perov theorem, as presented in [28], observe that, since $S^n \to \Theta$, $n \to \infty$, then there exists some $n \in \mathbb{N}$ such that $\|S^n\| = q < 1$, where assumed norm is the supremum norm. For such chosen n, (3.10) implies

$$d(f_1^n(u,y), f_1^n(v,y)) \leq S^n\left(d(u,v)\right), \quad u,v \in X, y \in Y,$$

so

$$d_\infty(f_1^n(u,y), f_1^n(v,y)) \leq q d_\infty(u,v), \quad u,v \in X, y \in Y,$$

where $d_\infty : X \times X \mapsto \mathbb{R}$ is a maximum metric defined with

$$d(u,v) = \max_{i=\overline{1,m}} d_i(u_i,v_i), \;\; u,v \in X.$$

Hence, Consequence 3.1 guarantees unique fixed point x^* of a mapping $f_1^n(\cdot,y) : X \mapsto X$ for any $y \in Y$. As in the proof of Theorem 3.6, x^* is also unique fixed point of $f_1(\cdot,y) : X \mapsto X$ for a fixed $y \in Y$. The rest of the proof would follow analogously as in [28].
As stated in this paper, Y could be any compact convex subset of a Banach space. This results is applied in solving systems of functional-differential equations such as:

$$\begin{aligned} x(t) &= \int_0^1 K(t,s,x(s),y(s))ds + g(t), \quad t \in [0,1], \\ y(t) &= \int_0^1 H(t,s,x(s),y(s),y(y(s)))ds, \quad t \in [0,1], \end{aligned}$$

where $x \in X$ and $y \in Y$, continuous mappings $K \in C\left([0,1] \times [0,1] \times \mathbb{R}^m \times [0,1], \mathbb{R}^m\right)$, $g \in C\left([0,1], \mathbb{R}^m\right)$ and $H \in C\left([0,1] \times [0,1] \times \mathbb{R}^m \times [0,1] \times [0,1], \mathbb{R}\right)$,

Under assumptions that codomain of H is contained in $[0,1]$, that H is a first coordinate Lipschitzian mapping with a constant L and K is a Perov generalized contraction, this system has at least one solution in $X \times Y$ for $X = C([0,1], \mathbb{R}^m) = \prod_{i=1}^m X_i$, $X_i = C[0,1]$, $i = \overline{1,m}$ and Y set of all Lipschitzian mappings on $C([0,1],[0,1])$ with a constant L. Observe that we could not use Banach theorem instead of Perov to obtain this conclusion due to the contractive condition for K.

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