IT4I

Teaching Formal Set Theory with Regard to Students' Comprehension

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Workshop on Modern Teaching, May 16, 2019

Co-funded by the
Erasmus+ Programme
of the European Union

TeComp

Situation

The task I've been given since 3 years ago:

Teach set theory ($\sim$15 $\times$ 90 minutes) to $\sim$20 students of

- 1st-year applied mathematics and, jointly,
- 4th-year mathematics in education

= NOT students of theoretical mathematics

The challenge:

- An abstract subject for non-theoreticians
  (not central to their field of study)
- Available Czech textbooks are purely technical, axiomatic
  (the last informal Czech textbook on set theory is from the 1940's and while good, it covers just a part of the syllabus)

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Goals and strategy

Aims:

1. Not to disgust the students by too abstract mathematics
   - Applied maths: one of their first university courses
   - Future teachers: one of their last math subjects
2. Get them (at least somewhat) interested in set theory
3. Teach them the basics of the subject
   (set operations, cardinals, ordinals, axioms)

Strategy:

- Arrive at interesting topics as soon as possible
- Motivate the concepts by intriguing notions
  (paradoxes, the hierarchy of infinities)
- Proceed from the familiar (number domains, geometric sets)
  to the abstract (cardinals, well-orderings, . . .)

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Implementation

- Reduce the syllabus (essentials only)
- Rearrange the subject matter (axioms last)
- Disregard set-theoretic purity (eg, prove the Schröder–Bernstein Theorem by a zig–zag construction using $\mathbb{N}$, rather than by the Tarski fixed-point theorem)
- Don't define concepts they think they know (ordered pairs, natural numbers)—make them precise later (towards the end of the course)
- Don't prove the claims they consider evident (eg, that there is no bijection between a finite and infinite set)—they can learn it later
- Focus mainly on the early developments in set theory (1874–1906)
- Invent both serious and funny (but illustrative) exercises

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The syllabus overhauled

Outline of the syllabus:

1. The notion of set, basic operations
2. Galilei's paradox ($\mathbb{N} \approx \mathbb{Z}$) $\Rightarrow$ inclusion vs subvalence, Schröder–Bernstein Theorem, the cardinalities of $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathscr{P}(\mathbb{R}), \ldots$
3. Cantor's Theorem, proper classes, cardinal numbers
4. Ordinal numbers, well-orderings, Zermelo's Theorem, choice
5. Axiomatic set theory, set-theoretic reconstruction of mathematical objects (Kuratowski's ordered pairs, von Neumann ordinals, . . .)

Topics sacrificed: transfinite induction (only sketched), Zorn's lemma, Zermelo's cumulative universe, inaccessible cardinals, metamathematics

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Example: The notion of set

Formally, *set* is a primitive notion implicitly defined by the axioms of ZFC

(too abstract: what motivates the axioms?)

I prefer starting from Cantor's 1895 informal definition of sets:

"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought, which are called elements of the set."

Later on, the vague definition is made more precise by adding several *set-theoretic principles* (extensionality, bivalence, well-foundedness, infinity, choice, cautious comprehension, CH/GCH, . . .)

Towards the end of the semester, these principles get expressed by 1st-order formulae and formulated as the axioms of ZFC

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Example: Cardinal numbers

Modern formal definition:

Cardinals = initial ordinals (ie, with no bijection onto smaller ordinals)

Disadvantage: Needs the notion of ordinals first
(which seems more complex than the basic idea of cardinality)

A more intuitive definition—by abstraction:

Cardinals = abstract objects assigned to sets, where two sets A, B are assigned the same cardinal iff a bijection $F\colon A \leftrightarrow B$ exists

Advantage: Can immediately introduce $\aleph_0$, $\mathfrak{c}$, $2^{\mathfrak{c}}$ and use them
(the modern definition is mentioned at the end of the course)

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Example: The beth-series of cardinals

Unlike most textbooks, I introduce the cardinal numbers $\beth_\alpha$ before $\aleph_\alpha$

Recall:

$$\beth_0 = |\mathbb{N}|, \quad \beth_1 = |\mathscr{P}(\mathbb{N})|, \quad \beth_2 = |\mathscr{P}(\mathscr{P}(\mathbb{N}))|, \quad \ldots$$

Advantages:

- $\beth_0, \beth_1, \beth_2$ are the cardinalities of the well-known sets $\mathbb{N}, \mathbb{R}, \mathscr{P}(\mathbb{R})$, while $\aleph_1$ is elusive without CH
- Transfinite steps in the $\beth$-sequence $(\beth_\omega, \beth_{\omega+1}, \ldots)$ motivate the need for transfinite ordinal numbers
- Provides another perspective on CH $(\aleph_1 = \beth_1)$ and GCH $(\aleph_\alpha = \beth_\alpha)$, after introducing ordinals and the $\aleph$-series

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Example: Ordinal numbers

- **Formal definition:** ordinals are transitive sets well-ordered by $\in$
  (technical, all intuition lost)
- **Definition by abstraction:** ordinals = types of well-orderings
  (more intuitive, but needs a good grasp of well-orders)
- **Definition by a generative principle** (the one I use):

  Every set of ordinals has an immediate successor ordinal.

Advantages:

- Can show the generation of the transfinite sequence of ordinals
  $(0, 1, 2, \ldots, \omega, \omega + 1, \ldots)$
- Immediate that ordinals form a proper class
  (if they formed a set, it'd have a successor)
- Easy to prove that they are well ordered, etc

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Example: Ordinal arithmetic

Intuition: transfinite *queues*

(queues have the required property of immediate successors)

Ordinals = numbers in a (transfinite) ticket queue

Ordinal addition = concatenating queues

Ordinal multiplication = each ticket number represents a sub-queue

Advantages:

- Intuitive representation of operations
- The difference between $1 + \omega$ and $\omega + 1$ immediately recognizable

(as tested with a 6-year old child)

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Exercises

(a) "Serious" exercises, to practice requisite skills

Examples:

- Find a bijection between the intervals $[0, 1]$ and $(0, 1)$
- Calculate: cardinals $(2 + \aleph_0)^2$, $2^{\aleph_0} + \aleph_0^2$, ordinals $(\omega + 1) \cdot 2$, $(\omega + 1)^2$
- Order by size: cardinals $\mathfrak{c}^3, 2^{\aleph_0}, 2^{\mathfrak{c}} + \mathfrak{c}^2$, ordinals: $3 + \omega$, $\omega + 2$, $\omega \cdot 2$

(b) "Funny" (but illuminative) word problems, to reinforce intuitions about transfinite numbers and operations

A classical example—Hilbert's hotel

A hotel with $\aleph_0$ single rooms is fully booked. Another guest arrives.

(a) Can the new guest be accommodated (by reassigning rooms)?
(b) A bus with further $\aleph_0$ guests arrives ...
(c) Then $\aleph_0$ buses, each with $\aleph_0$ guests ...

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Word problems on cardinalities

Some of the word problems devised for the course:

For a summer job, George picked apples in a plantation with $\aleph_0$ specially grafted apple trees, each with $\aleph_0$ branches, each of which bore $\aleph_0$ apples. For each apple picked he got 1 cent. On Monday he managed to pick all the apples in the plantation. (a) How might he have proceeded when picking, not to get stuck at a single tree? (b) How many euros did he earn?

On Tuesday, he worked in the Mathematical Supermarket, putting price tags to sets of real numbers. During his shift, he managed to tag all sets of reals. For each tag placed he earned 1 cent. How much money does he have now?

On Wednesday he accepted a bet with John for $2^{\mathfrak{c}}$ euros, and lost. How should he pay to John, to keep as much of his hard-earned money as possible?

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For a summer job, George picked apples in a plantation with $\aleph_0$ specially grafted apple trees, each with $\aleph_0$ branches, each of which bore $\aleph_0$ apples. For each apple picked he got 1 cent. On Monday he managed to pick all the apples in the plantation. (a) How might he have proceeded when picking, not to get stuck at a single tree? (b) How many euros did he earn?

On Tuesday, he worked in the Mathematical Supermarket, putting price tags to sets of real numbers. During his shift, he managed to tag all sets of reals. For each tag placed he earned 1 cent. How much money does he have now?

On Wednesday he accepted a bet with John for $2^{\mathfrak{c}}$ euros, and lost. How should he pay to John, to keep as much of his hard-earned money as possible?

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Word problems on proper classes

A generative principle similar to that for ordinals:

Breeding goblins are defined as such creatures that each set of breeding goblins breeds another breeding goblin, different from any other.

(a) Demonstrate that there exists a breeding goblin.

(b) Show that in fact there is a proper class of breeding goblins.

(c) How comes that we proved the existence of breeding goblins?

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Word problems on ordinal numbers

At the Championship of Cantor's Paradise, Jane won the third place and Emma ended up $\omega + 3$ positions behind Jane. What was Emma's position?

At the same competition, Lilly placed $(\omega^2 + \omega + 4)$-th.

(a) Which position Lilly was behind Emma?

(b) How many competitors were between Emma and Jane?

To celebrate the opening of the $(\omega + 4)$-th Pandimensional Olympics, the athletes formed a grid with $\omega + 4$ files and $\omega + 4$ rows. After the ceremony they left through the main gate of the Infinite Stadium, row by row. What number in line was the last athlete?

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Goals achieved

Topics covered: set operartions, cardinals, ordinals, CH, AC, proper classes, main theorems (Cantor, Schröder–Bernstein, Zermelo), axioms

Skills acquired:

- Determining the cardinalities of sets from common mathematics ($\mathbb{N} \times \mathbb{N}$, $\mathbb{R} \smallsetminus \mathbb{Q}$, geometric figures, the Cantor set, etc)
- Constructing bijections and injections between sets
- Simple calculations with cardinal and ordinal numbers

Success in students' comprehension?

- Some success apparent already in the lecture room (exercises solved)
- We'll see tomorrow (the 1st exam term)

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Past and future work

This year was the third run of the course. Iterative improvements:

- 1st year: definitions and proofs reworked, BUT: too many topics in the wrong order, too many details $\Rightarrow$ too little room for exercises
- 2nd year: handouts, fewer topics BUT still too many, still in wrong order, still too little room for exercises
- 3rd year: at last sufficiently pruned, topics in the right order, more time for exercises

Possible improvements for next years:

- Turning handouts into a course book
- Videotaping lectures, to be watched as homework

  (to save even more time for exercises and perhaps more topics)

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